GRAVITATIONAL LENSING AND THE EINSTEIN RING

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We saw that when light from a distant source passes near to a mass M, the light's path is bent by an angle of

$$\delta = 2\left|\phi_0\right| = \frac{4GM}{r_c}$$

where r_c is the distance of closest approach to the mass. If the distant source is almost directly behind the mass (as viewed by the observer) then photons can pass by the mass on all sides. Suppose we consider a photon that passes above (OK, 'above' is relative, but just pick a side as 'up' and assume the photon passes the object on that side) the mass. Its path is bent downwards so the source appears to be further up in the sky than if the mass were not there. Similarly, if a photon passes below the mass its path is bent upwards and the source appears further down in the sky. Thus from these two paths, we get two separate images of the same source, one on either side of the mass. If the source is directly behind the mass, the same argument shows that an image is produced on all sides of the mass, creating a ring known as the *Einstein ring*. The splitting of a source image into two or more apparent images (or a ring) is known as *gravitational lensing*.

If the source isn't exactly behind the mass, we get two images, one on either side of the mass. To work out where these images appear, we need a few definitions. Figure 13.4 in Moore's book will be helpful here (it's too complicated for me to try to reproduce here, but I'll describe the main quantities and you can try drawing the diagram yourself as we go along).

First, we'll refer to the distant source as S and the mass that causes the lensing as L, and the location of the observer as O. Draw the straight line through O and L, and then another straight line through O and S. The latter line represents where S would appear if L were not there. The angle between OS and OL we will call β, which is assumed to be a small angle (that is, L and S would appear very close to each other in the sky in the absence of lensing).

Now draw a line from S to some point L' close to, but above, L. This line represents a photon leaving S and grazing the lensing mass L. From its closest approach to L, draw another line from L' to the observer at O. The line $L'O$ gives the apparent position of S as seen by O. The smallest angle between SL' and $L'O$ is δ, the deflection angle. Finally, define the angle between $L'O$ and OL as θ. (Note: Moore's diagram can be a bit misleading in that it looks like SL' and LO are parallel, which would seem to imply that $\delta = \theta$. This is not necessarily the case, since we can draw many different photon paths starting from S and coming near to L, so the line SL' can take on several orientations, while the line OL is fixed.)

Under the small angle approximation, we will take all sines and tangents to be equal to the angle itself, and all cosines to be 1. Under this assumption, the distance of closest approach is equal to the impact parameter b, which is the distance $L'O$ times $\sin\theta$. For small angles $|L'O|\cos\theta = |L'O| = |LO| \equiv D_L$, so

$$b = D_L\theta$$

Now suppose we extend the line OL until it has the same length D_S as the distance to S. Then the distance between the endpoint of that extended line and S itself is $D_S\beta$. Now extend the line OL' until it too has length D_S (remember all these operations are assuming the angles β, δ and θ are very small). Then the distance from S to the endpoint of this extended line is $D_{LS}\delta$ where D_{LS} is the distance from L to S and δ is the deflection angle.

The total distance between the image of L and the image of S is $D_S\theta$ since θ is the angle between these two images as seen by the observer. But this must be the sum of the two distances in the previous paragraph, so

$$D_S\theta = D_{LS}\delta + D_S\beta$$

From above, we can write

$$\delta = \frac{4GM}{r_c} = \frac{4GM}{b} = \frac{4GM}{D_L\theta}$$

so

$$\begin{aligned} D_S\theta &= D_{LS}\frac{4GM}{D_L\theta} + D_S\beta \\ \theta^2 - \beta\theta - D_{LS}\frac{4GM}{D_L D_S} &= 0 \\ \theta^2 - \beta\theta - \theta_E^2 &= 0 \end{aligned}$$

where

$$\theta_E \equiv \sqrt{D_{LS}\frac{4GM}{D_L D_S}}$$

If $\beta = 0$, the source is directly behind the lens and $\theta = \pm\theta_E$ so that θ_E is the angle subtended by the radius of the Einstein ring. For $\beta \neq 0$, the two lensed images of the source will then appear at

$$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

If we take the plus sign, we have $\theta_+ > \theta_E$ so this image appears outside the Einstein ring.

Taking the minus sign, we have $\theta_- < 0$. From the original quadratic equation then:

$$\begin{aligned} \theta^2 + \beta\,|\theta| &= \theta_E^2 \\ \theta^2 &< \theta_E^2 \end{aligned}$$

so the second image appears inside the Einstein ring.

If the source S subtends a non-zero angle as viewed by the observer (that is, it's an extended object such as a galaxy rather than a point source like a star), then β varies over the extent of the source. Since the relation between β and θ is non-linear, the apparent angle subtended by the lensed image is not directly proportional to β. We can get an estimate of the apparent angle subtended by the image by taking the differential of θ:

$$\Delta\theta_{\pm} = \frac{1}{2}\left(1 \pm \frac{\beta}{\sqrt{\beta^2 + 4\theta_E^2}}\right)\Delta\beta$$

Since $\frac{\beta}{\sqrt{\beta^2+4\theta_E^2}} < 1$, $0 < \frac{\Delta\theta}{\Delta\beta} < 1$, which means that the angle subtended by the image is always less than the angle subtended by the source in the absence of the lensing mass. In other words, the image is always compressed, with the inner image being more compressed than the outer one.

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