GRAVITATIONAL REDSHIFT FROM THE KILLING VECTOR

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Post date: 27 April 2023.

We show here an alternative derivation of the gravitational redshift in Schwarzschild metric. We begin with the quantum mechanical energy of a photon, given by

$$E = \hbar\omega \tag{1}$$

where $\hbar$ is Planck's constant divided by 2π and ω is the frequency of the photon.

The redshift is a difference is the observed frequency of a photon between its point of emission and its point of detection. As we're usually interested in detecting the redshift of a distant object such as a star or galaxy, we can take the point of emission to be the distance from the centre of the star and the point of detection as essentially infinity.

From the relation between energy and velocity of observer we know that

$$E = -\boldsymbol{p}\cdot\boldsymbol{u}_{\text{obs}} \tag{2}$$

We also know that the four-velocity satisfies the identity

$$\boldsymbol{u}_{\text{obs}}\cdot\boldsymbol{u}_{\text{obs}} = g_{\mu\nu}u^{\mu}_{\text{obs}}(r)\cdot u^{\nu}_{\text{obs}}(r) \tag{3}$$

$$= -1 \tag{4}$$

where r is the distance from the star.

In our redshift experiment, the observer is an astronomer in their rest frame on Earth at, essentially, $r=\infty$. If we assume that the star emitting the photon is at rest relative to Earth, and is at a distance $r=R$, we can apply the Schwarzschild metric, which is (in geometric units with $G=c=1$)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right) \tag{5}$$

As both the source and observer are at rest, $u^{\mu}=0$ for $\mu=1,2,3$, so we have

$$\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{u}_{\text{obs}} = g_{tt}\left[u_{\text{obs}}^{t}(r)\right]^{2} \tag{6}$$
$$= -\left(1-\frac{2M}{r}\right)\left[u_{\text{obs}}^{t}(r)\right]^{2} \tag{7}$$
$$= -1 \tag{8}$$

which gives

$$u_{\text{obs}}^{t}(r) = \left(1-\frac{2M}{r}\right)^{-1/2} \tag{9}$$

Since the Schwarzschild metric is independent of t, a Killing vector associated with this metric is

$$\boldsymbol{\xi} = (1,0,0,0) \tag{10}$$

We can therefore write the four-velocity as

$$\boldsymbol{u}_{\text{obs}}(r) = \left(1-\frac{2M}{r}\right)^{-1/2}\boldsymbol{\xi} \tag{11}$$

From 2, the energy of a photon at distance r from the star is

$$E(r) = -\left(1-\frac{2M}{r}\right)^{-1/2}\boldsymbol{p}\cdot\boldsymbol{\xi} \tag{12}$$

However, we know that the quantity $\boldsymbol{p}\cdot\boldsymbol{\xi}$ is conserved, so its value is independent of r. To calculate the redshift, all we need is the ratio of energies at two distances, so the $\boldsymbol{p}\cdot\boldsymbol{\xi}$ term cancels out. Comparing $r = R$ and $r = \infty$, we have

$$\frac{E(R)}{E(\infty)} = \left(1-\frac{2M}{R}\right)^{-1/2} \tag{13}$$

or, in terms of photon frequency

$$\frac{\omega_{*}}{\omega_{\text{obs}}} = \left(1-\frac{2M}{R}\right)^{-1/2} \tag{14}$$

or

$$\omega_{\text{obs}} = \left(1-\frac{2M}{R}\right)^{1/2}\omega_{*} \tag{15}$$

where ω_{*} is the frequency of the photon emitted by the star and ω_{obs} is the frequency measured on Earth. The redshift can be obtained, for example, by measuring the shift in lines in the star's spectrum. If the star's mass is known, then its distance can be determined.

The observed frequency is less than the emitted frequency, which means that its wavelength is longer, so that it's a redshift (as opposed to a blueshift). In terms of wavelength, the redshift formula is the inverse of that for frequency:

$$\lambda_{\text{obs}} = \left(1 - \frac{2M}{R}\right)^{-1/2} \lambda_* \tag{16}$$