

## GRAVITOELECTRIC AND GRAVITOMAGNETIC DENSITIES

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We can now have a look at the equation of motion for a particle moving in the weak field, slow velocity limit. We begin with the geodesic equation

$$\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0 \quad (1)$$

where a dot indicates a derivative with respect to proper time  $\tau$ . Using the chain rule we can write the first term as

$$\frac{d^2 x^m}{d\tau^2} = \frac{d}{d\tau} \left( \frac{dx^m}{d\tau} \right) \quad (2)$$

$$= \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dx^m}{d\tau} \right) \quad (3)$$

$$= \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dt}{d\tau} \frac{dx^m}{dt} \right) \quad (4)$$

$$= u^t \frac{dt}{d\tau} \frac{d^2 x^m}{dt^2} + v^m \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dt}{d\tau} \right) \quad (5)$$

$$= u^t u^t \frac{d^2 x^m}{dt^2} + v^m \frac{d}{d\tau} \left( \frac{dt}{d\tau} \right) \quad (6)$$

$$= u^t u^t \frac{d^2 x^m}{dt^2} + v^m \frac{du^t}{d\tau} \quad (7)$$

where

$$v^m \equiv \frac{dx^m}{dt} \quad (8)$$

Now suppose the particle is moving slowly so that both  $u^i$  (where  $i$  is a spatial index) and  $v^i$  are small, so we can ignore second order terms. In this approximation,  $u^t \approx 1$  so we have from 1 with  $m = t$

$$\frac{du^t}{d\tau} = -\Gamma_{ij}^t \dot{x}^j \dot{x}^i \quad (9)$$

$$= -\Gamma_{ij}^t u^j u^i \quad (10)$$

$$\approx -\Gamma_{tt}^t u^t u^t \quad (11)$$

$$= -\Gamma_{tt}^t \quad (12)$$

Therefore

$$v^m \frac{du^t}{d\tau} \approx -v^m \Gamma_{tt}^t \quad (13)$$

The Christoffel symbols in terms of the metric are

$$\Gamma_{ij}^m = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (14)$$

For a stationary field, all time derivatives are zero, so

$$\Gamma_{tt}^t = -\frac{1}{2} g^{tl} \partial_l g_{tt} \quad (15)$$

In the weak field limit, we are using a metric that is perturbed from the flat space metric:

$$g_{ij} = \eta_{ij} + h_{ij} \quad (16)$$

$$g^{ij} = \eta^{ij} - h^{ij} \quad (17)$$

So to first order in  $h_{ij}$ :

$$\Gamma_{tt}^t = -\frac{1}{2} (\eta^{tl} - h^{tl}) \partial_l (\eta_{tt} + h_{tt}) \quad (18)$$

$$= -\frac{1}{2} \eta^{tt} \partial_t h_{tt} + \frac{1}{2} h^{tl} \partial_l \eta_{tt} \quad (19)$$

$$= 0 \quad (20)$$

since all time derivatives are zero in the stationary field, and all derivatives of the flat space metric are zero as well. Therefore in this approximation 7 becomes

$$\frac{d^2 x^m}{d\tau^2} = u^t u^t \frac{d^2 x^m}{dt^2} = \frac{d^2 x^m}{dt^2} \quad (21)$$

and the geodesic equation 1 becomes

$$\frac{d^2 x^m}{dt^2} = -\Gamma_{ij}^m u^j u^i \quad (22)$$

$$\approx -\Gamma_{tt}^m u^t u^t - \Gamma_{it}^m u^t u^i - \Gamma_{tj}^m u^t u^j \quad (23)$$

$$= -\Gamma_{tt}^m - 2\Gamma_{it}^m u^i \quad (24)$$

where to get the last line we used the symmetry of the Christoffel symbols:  $\Gamma_{it}^m = \Gamma_{ti}^m$ .

In the weak field limit, the Christoffel symbols become

$$\Gamma_{ij}^m = \frac{1}{2}\eta^{ml} (\partial_j h_{il} + \partial_i h_{lj} - \partial_l h_{ji}) \quad (25)$$

In this approximation

$$\Gamma_{tt}^m = \frac{1}{2}\eta^{ml} (\partial_t h_{tl} + \partial_t h_{lt} - \partial_l h_{tt}) \quad (26)$$

$$= -\frac{1}{2}\eta^{ml} \partial_l h_{tt} \quad (27)$$

since time derivatives are zero for a stationary field.

Also

$$2\Gamma_{it}^m u^i = u^i \eta^{ml} (\partial_t h_{il} + \partial_i h_{lt} - \partial_l h_{it}) \quad (28)$$

$$= u^i \eta^{ml} (\partial_i h_{lt} - \partial_l h_{it}) \quad (29)$$

Putting it all together, the geodesic equation 24 becomes

$$\frac{d^2 x^m}{dt^2} = \eta^{ml} \left[ \frac{1}{2} \partial_l h_{tt} + u^i (\partial_l h_{it} - \partial_i h_{lt}) \right] \quad (30)$$

Remember that all indices (except  $t$ ) range over only spatial components. In the weak field limit, the perturbations to the metric are given by

$$h_{jm} = 2G \int \frac{2T_{jm} - \eta_{jm} T}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}' \quad (31)$$

The numerator in the integrand is analogous to charge density in electromagnetism. The first term in 30 results from the density

$$\rho_g \equiv 2T_{tt} - \eta_{tt} T \quad (32)$$

which is called the *gravitoelectric energy density*, since it gives rise to an acceleration of the particle that does not depend on the particle's velocity, in a similar way to an electric field accelerating a charge independently of the charge's velocity.

The second term in 30 results from the density

$$\Pi_j \equiv T_{tj} - \frac{1}{2}\eta_{tj}T \quad (33)$$

which is called the *gravitomagnetic current density* since it contributes to an acceleration that is proportional to the particle's velocity, in a similar way to the Lorentz force arising from magnetism.

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