HIGHER ORDER DERIVATIVES ARE NOT TENSORS

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We've seen that the tangent to a curve is a contravariant tensor (vector, actually), since it transforms according to

$$\frac{dx'^{\alpha}}{du}=\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{dx^{\mu}}{du} \tag{1}$$

Also, the first derivative of a function is a covariant tensor, as it transforms according to

$$\frac{\partial g}{\partial x'^{\alpha}}=\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial g}{\partial x^{\mu}} \tag{2}$$

You might think that higher order derivatives are also tensors, but this turns out not to be the case. If we take the derivative of the last equation with respect to another of the primed coordinates x'^{c}, we get

$$\begin{aligned}\frac{\partial^2 g}{\partial x'^{\alpha}\partial x'^{\gamma}} &= \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha}\partial x'^{\gamma}}\frac{\partial g}{\partial x^{\mu}}+\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial^2 g}{\partial x^{\mu}\partial x'^{\gamma}} && (3)\\ &= \frac{\partial^2 x^{\mu}}{\partial x'^{\alpha}\partial x'^{\gamma}}\frac{\partial g}{\partial x^{\mu}}+\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\gamma}}\frac{\partial^2 g}{\partial x^{\mu}\partial x^{\nu}} && (4)\end{aligned}$$

where we used the chain rule on the second term. In order for $\frac{\partial^2 g}{\partial x^{\mu}\partial x^{\nu}}$ to transform like a tensor, the first term in the last line would have to be zero, but it's clearly not, in general.

Higher order derivatives will also have extraneous terms, so it is only first derivatives that behave as tensors.

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