HYPERBOLIC COORDINATES IN FLAT SPACE

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The strange behaviour of Schwarzschild coordinates at the event horizon $r = 2GM$ might seem to be a result of the fact that we are describing a curved space-time, but in fact we can define coordinates in flat space-time that behave in a similar fashion. If we use rectangular coordinates in flat space-time, the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

Suppose we now introduce replacements for t and x, as defined by

$$t(T,R) \equiv \pm b\sqrt{2R-1}\sinh T \tag{2}$$

$$x(T,R) \equiv \pm b\sqrt{2R-1}\cosh T \tag{3}$$

for $R > \frac{1}{2}$ and by

$$t(T,R) \equiv \pm b\sqrt{1-2R}\cosh T \tag{4}$$

$$x(T,R) \equiv \pm b\sqrt{1-2R}\sinh T \tag{5}$$

for $R < \frac{1}{2}$.

Here, R and T are dimensionless and b is a positive constant with dimensions of length. We take either the plus sign in both definitions or the minus sign in both.

The constant curves in this new system can be found as follows. First, we take R to be constant. Then for $R > \frac{1}{2}$

$$x^2 - t^2 = b^2(2R-1)\left(\cosh^2 T - \sinh^2 T\right) = b^2(2R-1) \tag{6}$$

and for $R < \frac{1}{2}$

$$t^2 - x^2 = b^2(1-2R)\left(\cosh^2 T - \sinh^2 T\right) = b^2(1-2R) \tag{7}$$

This is the equation of a hyperbola in $x-t$ coordinates, with asymptotes of $t = \pm x$. If $R > \frac{1}{2}$, $x^2 > t^2$ and the hyperbola cuts the x axis at $x = \pm b\sqrt{2R-1}$ with the curves opening to the left and right. If $R < \frac{1}{2}$, the hyperbola cuts the t axis at $x = \pm b\sqrt{2R-1}$ with the curves opening to the top and bottom.

If T is a constant, then for $R > \frac{1}{2}$

$$\frac{t}{x} = \tanh T = \text{constant} \tag{8}$$

so the lines of constant T are straight lines with slope $\tanh T$. Since $|\tanh T| < 1$ these lines all have slopes between -1 and $+1$. For $R < \frac{1}{2}$ we get

$$\frac{t}{x} = \coth T = \text{constant} \tag{9}$$

and these lines all have slopes greater than 1 (or less than -1). Note that the line for $T = +\infty$ is the line $t = x$ which coincides with one of the solutions for constant R when $R = \frac{1}{2}$, since in the latter case, $x^2 - t^2 = 0$, so $t = \pm x$.

In the new coordinates, the metric can be found by applying the usual chain rule. For $R > \frac{1}{2}$

$$dt = \pm \frac{b}{\sqrt{2R-1}} \sinh T dR \pm b\sqrt{2R-1} \cosh T dT \tag{10}$$

$$dx = \pm \frac{b}{\sqrt{2R-1}} \cosh T dR \pm b\sqrt{2R-1} \sinh T dT \tag{11}$$

So:

$$\begin{aligned} -dt^2 + dx^2 = {} & \frac{b^2}{2R-1}\left(\cosh^2 T - \sinh^2 T\right) dR^2 - b^2 (2R-1)\left(\cosh^2 T - \sinh^2 T\right) dT^2 \\ & - b^2 \cosh T \sinh T dR dT + b^2 \cosh T \sinh T dR dT \qquad (12) \\ = {} & \frac{b^2}{2R-1} dR^2 - b^2 (2R-1) dT^2 \qquad (13) \end{aligned}$$

For $R < \frac{1}{2}$ the process is very similar:

$$dt = \pm \frac{b}{\sqrt{1-2R}} \cosh \sinh T dR \pm b\sqrt{1-2R} \sinh T dT \tag{14}$$

$$dx = \pm \frac{b}{\sqrt{1-2R}} \sinh T dR \pm b\sqrt{1-2R} \cosh T dT \tag{15}$$

So:

$$
\begin{aligned}
-dt^2+dx^2 &= -\frac{b^2}{1-2R}\left(\cosh^2 T-\sinh^2 T\right)dR^2+b^2\left(1-2R\right)\left(\cosh^2 T-\sinh^2 T\right)dT^2 \\
&\quad -b^2\cosh T\sinh T\,dRdT+b^2\cosh T\sinh T\,dRdT \qquad (16)\\
&= -\frac{b^2}{1-2R}dR^2+b^2\left(1-2R\right)dT^2 \qquad (17)\\
&= \frac{b^2}{2R-1}dR^2-b^2\left(2R-1\right)dT^2 \qquad (18)
\end{aligned}
$$

Thus the metric is the same in all regions of the space-time. The full metric is then

$$ds^2=\frac{b^2}{2R-1}dR^2-b^2\left(2R-1\right)dT^2+dy^2+dz^2 \qquad (19)$$

For $R>\frac{1}{2}$, the T component of the metric is negative and thus represents time, and the R component is positive so represents space. For $R<\frac{1}{2}$ the reverse is true, so R becomes the time coordinate.

At this point, it's useful to have a look at the geodesic equation for this metric. The geodesic equation is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \qquad (20)$$

The T component (with the index $a=T$) gives

$$\frac{d}{d\tau}\left(g_{TT}\frac{dT}{d\tau}\right)=0 \qquad (21)$$

$\partial g_{ij}/\partial T=0$ for all metric components, since none of them depends on T. Integrating this, we get

$$\frac{dT}{d\tau}=\frac{-e}{g_{TT}}=\frac{e}{b^2\left(2R-1\right)} \qquad (22)$$

where $-e$ is a constant of integration. From here, we can use the metric equation 19 with no motion in the y or z directions to say

$$d\tau^2 = -ds^2 = -\frac{b^2}{2R-1}dR^2 + b^2(2R-1)\,dT^2 \tag{23}$$

$$1 = -\frac{b^2}{2R-1}\left(\frac{dR}{d\tau}\right)^2 + b^2(2R-1)\left(\frac{dT}{d\tau}\right)^2 \tag{24}$$

$$1 = -\frac{b^2}{2R-1}\left(\frac{dR}{d\tau}\right)^2 + \frac{e^2}{b^2(2R-1)} \tag{25}$$

$$\frac{dR}{d\tau} = \pm\sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}} \tag{26}$$

$$\frac{d^2R}{d\tau^2} = \pm\frac{1}{2}\left(\frac{e^2}{b^4} - \frac{2R-1}{b^2}\right)^{-1/2}\left(-\frac{2}{b^2}\frac{dR}{d\tau}\right) \tag{27}$$

$$= -\frac{1}{b^2} \tag{28}$$

Thus the second derivative $\frac{d^2R}{d\tau^2}$ is a negative constant which means that

$$R(\tau) = -\frac{\tau^2}{2b^2} + R_1\tau + R_0 \tag{29}$$

for constants R_1 and R_0. Since we know the first derivative, we must have

$$\frac{dR}{d\tau} = \pm\sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}} \tag{30}$$

$$-\frac{\tau}{b^2} + R_1 = \pm\frac{1}{b^2}\sqrt{e^2 + \tau^2 - 2b^2R_1\tau - 2b^2R_0 + b^2} \tag{31}$$

$$\frac{\tau^2}{b^4} - 2\frac{R_1}{b^2}\tau + R_1^2 = \frac{1}{b^4}\left(e^2 + \tau^2 - 2b^2R_1\tau - 2b^2R_0 + b^2\right) \tag{32}$$

From the constant term on both sides, we have

$$R_1^2 = \frac{1}{b^4}\left(e^2 - 2b^2R_0 + b^2\right) \tag{33}$$

To proceed further, we need to specify some initial conditions, so we'll come back to this in a minute.

From $\frac{dR}{d\tau}$ and $\frac{dT}{d\tau}$ we can get $\frac{dR}{dT}$:

$$\frac{dR}{dT} = \frac{dR/d\tau}{dT/d\tau} \tag{34}$$

$$= \pm\frac{b^2(2R-1)}{e}\sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}} \tag{35}$$

This can be integrated (using Maple) to get

$$T=\pm\frac{1}{2}\ln\left|\frac{\sqrt{b^2-2b^2R+e^2}-e}{\sqrt{b^2-2b^2R+e^2}+e}\right| \tag{36}$$

Note the absolute value in the logarithm: Maple does not show this, but it's important in what follows.

Now for the initial conditions. Suppose we start with an object at rest at $R=1$ when $T=\tau=0$. Then we must have $\frac{dR}{dT}(R=1)=0$, which we get

$$\frac{e^2}{b^4}-\frac{2\times1-1}{b^2}=0 \tag{37}$$

$$e^2=b^2 \tag{38}$$

$$e=\pm b \tag{39}$$

We must then have $R(\tau=0)=1$, from which we get $R_0=1$ from 29 and $R_1=0$ from 33 (with $e^2=b^2$). Therefore

$$R(\tau)=-\frac{\tau^2}{2b^2}+1 \tag{40}$$

$$\tau=\sqrt{2}b\sqrt{1-R} \tag{41}$$

$$T=\frac{1}{2}\ln\left|\frac{\sqrt{2-2R}-1}{\sqrt{2-2R}+1}\right| \tag{42}$$

We've taken positive square roots in both cases since both time coordinates increase from zero. Thus the total *proper* time τ required for the object to move from $R=1$ to $R=\frac{1}{2}$ is $\tau=b$ but the coordinate time T is infinite (since we get a $\ln 0$ term when $R=\frac{1}{2}$). If we look at this situation in the original t,x coordinates, the object starts at $(t,x)=(0,b)$ (using the transformation equations at the top of this post). Since it starts at rest and there are no forces, it remains at rest at $x=b$. When $R=\frac{1}{2}$, we have arrived at a point on the line $t=x$, so the object's coordinates are now $(t,x)=(b,b)$ and since $t=\tau$ for an object at rest, this is consistent with our earlier calculation.

As we mentioned earlier, if $R<\frac{1}{2}$, R is a time coordinate since its metric component is negative. If we look at the top sector where $R<\frac{1}{2}$, we see this corresponds to the region inside the light cone for an object at $(x,t)=(0,0)$. Thus the object is permitted to visit any point within that light cone but it cannot move outside the cone. Further, the object's world line must always have a tangent that is greater than 1 (that is, $dt/dx>1$ for every point on the world line). The hyperbolas defined by 7 are further up the t axis for smaller values of R (note that R can become negative here, unlike the radial coordinate in the Schwarzschild metric), so in this region, we must take the minus sign in the expression for $dR/d\tau$, so that

$$\frac{dR}{d\tau} = -\sqrt{\frac{e^2}{b^4} - \frac{2R-1}{b^2}} \tag{43}$$

Thus when R is a time coordinate, it steadily decreases.

For photons, $ds^2 = 0$ and the metric states (assuming $dy = dz = 0$)

$$0 = \frac{b^2}{2R-1}dR^2 - b^2(2R-1)\,dT^2 \tag{44}$$

$$\frac{dR}{dT} = \pm(2R-1) \tag{45}$$

When $R = \frac{1}{2}$, $dR/dT = 0$ so in these coordinates, a photon appears to be 'at rest'.

The argument given in Moore for massive particles goes like this. If we take a particle at rest, then $dR = dy = dz = 0$ so the metric becomes

$$ds^2 = \frac{b^2}{2R-1}dR^2 - b^2(2R-1)\,dT^2 + dy^2 + dz^2 = -b^2(2R-1)\,dT^2 \tag{46}$$

At $R = \frac{1}{2}$, the last term is zero, so $ds^2 = 0$, which can be true only for light-like particles. However, this seems to be a bit of a fudge since in taking $dR = 0$ *and* $R = \frac{1}{2}$, the term $\frac{b^2}{2R-1}dR^2$ becomes zero divided by zero, so how can we say this is zero?