# HYPERBOLIC MOTION AND ACCELERATION

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In special relativity, the invariant interval $\Delta s^2$ in a system with one space dimension and one time dimension is given by

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 \quad (1)$$

The world line of an object is a curve plotted on a $tx$ diagram that shows the object's position relative to an observer in the stationary lab frame. Since no object can travel faster than light, the tangent to a world line must have a slope greater than 1 (or less than $-1$ if it is travelling in the $-x$ direction) at all points on the curve. A photon's world line is a straight line with a slope of $\pm 1$.

If we take one event to be at the origin so that $t_0 = 0$ and $x_0 = 0$ then we can write 1 as

$$x^2 - t^2 = \pm r^2 \quad (2)$$

where we've replaced the invariant $\Delta s^2$ by the constant $\pm r^2$, which can be positive or negative. This is the equation of a hyperbola with the lines $t = \pm x$ as asymptotes. If we take the constant to be $+r^2$, then we have

$$t^2 - x^2 = -r^2 \quad (3)$$

which is a hyperbola that crosses the $x$ axis at the point $(t,x) = (0,r)$ (we're considering only the positive branch of the hyperbola).

The proper time experienced by an object in its own rest frame is the arc length along the object's world line. That is, the proper time is the integral of $-\Delta s^2 = \Delta t^2 - \Delta x^2$ along the world line. Note that this arc length is calculated using the Minkowski metric (where $\Delta x^2$ has a minus sign and $\Delta t^2$ has a plus sign) rather than the Euclidean metric (with $\Delta s^2 = \Delta x^2 + \Delta y^2$ in a typical $xy$ plot) that is used in 'normal' geometry.

We consider here the behaviour of an object moving along a hyperbolic trajectory given by 3 with $r = 1$, so we have

$$t^2 - x^2 = -1 \quad (4)$$

In Fig. 1, this is the red curve.

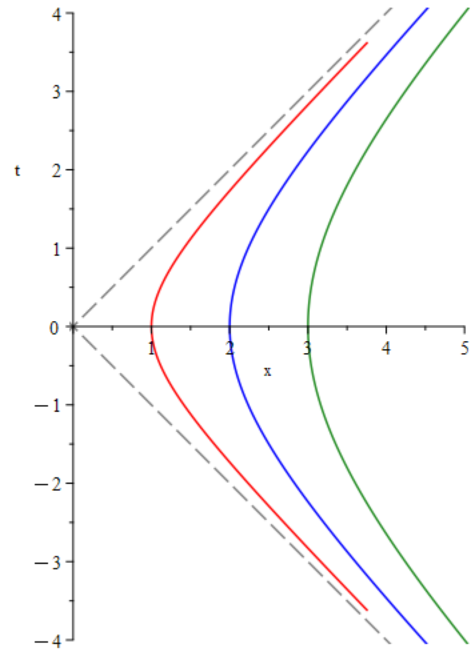


FIGURE 1. Hyperbolic motion. The curves correspond to $r = 1$ (red), $r = 2$ (blue) and $r = 3$ (green). Along each curve, the acceleration is constant, but it is different for each curve. The asymptotes are shown as grey dashed lines, and correspond to a path followed by a light ray.

First, we note that this hyperbola is an acceptable world line, since its tangent has a magnitude greater than 1 everywhere, so the object is always moving less than the speed of light. Second, the speed of the object starts out (at large negative values of $t$) to be large and in the $-x$ direction, reaching zero at $t = 0$ (where the tangent is vertical) and then speeding up again but this time in the $+x$ direction. That is, the object is accelerating in the $+x$ direction.

A hyperbola can be written in parametric form using (not surprisingly) hyperbolic functions. That is, we have

$$\begin{aligned} t &= \sinh \omega \\ x &= \cosh \omega \end{aligned} \tag{5}$$

where $\omega$ is the parameter which varies from $-\infty$ to $+\infty$. Using the identity $\cosh^2 \omega - \sinh^2 \omega = 1$, we see that this parametric form satisfies the original equation 4 for the hyperbola.

We now want to find the proper time for an object moving along this world line. We have (using the derivatives of the hyperbolic functions $\frac{d\cosh\omega}{d\omega} = \sinh\omega$ and $\frac{d\sinh\omega}{d\omega} = \cosh\omega$):

$$\begin{aligned}\Delta\tau^2 &= \Delta t^2 - \Delta x^2 &(6)\\ &= \left(\cosh^2\omega - \sinh^2\omega\right)\Delta\omega^2 &(7)\\ &= \Delta\omega^2 &(8)\end{aligned}$$

Thus the hyperbola's parameter $\omega$ is actually the same thing as the proper time $\tau$. Thus the proper time between two events parameterized by the two values $\omega \in [\omega_0, \omega_1]$ is just $\Delta\tau = \omega_1 - \omega_0$.

We can generalize this result by considering an arbitrary value for $r$ in 3. In that case, 5 becomes

$$\begin{aligned}t &= r\sinh\omega\\ x &= r\cosh\omega\end{aligned} \qquad (9)$$

and the proper time is now

$$\begin{aligned}\Delta\tau^2 &= \Delta t^2 - \Delta x^2 &(10)\\ &= r^2\left(\cosh^2\omega - \sinh^2\omega\right)\Delta\omega^2 &(11)\\ &= r^2\Delta\omega^2 &(12)\end{aligned}$$

In this case, the parameter $\omega$ is

$$\omega = \frac{\tau}{r} \qquad (13)$$

We can now calculate the acceleration of an object whose world line is the given hyperbola. The acceleration is the second derivative of the coordinates with respect to proper time, so we have

$$\begin{aligned}a_x &= \frac{d^2x}{d\tau^2} = \frac{1}{r}\cosh\frac{\tau}{r} &(14)\\ a_t &= \frac{d^2t}{d\tau^2} = \frac{1}{r}\sinh\frac{\tau}{r} &(15)\end{aligned}$$

The magnitude of the acceleration is (remember to use the Minkowski metric!)

$$|a|^2 = a_x^2 - a_t^2 \tag{16}$$
$$= \frac{1}{r^2}\left(\cosh^2\frac{\tau}{r} - \sinh^2\frac{\tau}{r}\right) \tag{17}$$
$$= \frac{1}{r^2} \tag{18}$$

Since $r$ is a constant for any given hyperbola, the acceleration is a constant, and its magnitude $|a| = \frac{1}{r}$ decreases with increasing minimum distance $r$ from the origin. Thus each curve in Fig. 1 is an object with constant acceleration, but the acceleration is different for each curve. The speed of the object is given by the inverse slope of the tangent line, so it never exceeds the speed of light. Thus an object can accelerate forever without its speed exceeding $c$.

We've done all these calculations using relativistic units, so $c = 1$. To restore 'normal' units of acceleration, we remember that its units are $[\text{distance}] \times [\text{time}]^{-2}$ so we can get these units by multiplying by $c^2$:

$$|a| = \frac{c^2}{r} \tag{19}$$

To be on a hyperbolic path with an acceleration of $|a| = g = 9.8$ m s$^{-2}$, we would need to have a minimum distance of

$$r = \frac{c^2}{9.8} \approx 9.2 \times 10^{15} \text{ m} \tag{20}$$

This is very nearly 1 light year ($9.46 \times 10^{15}$ m).