HYPERBOLIC PLANE

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The hyperbolic plane is defined by the metric

$$dS^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) \tag{1}$$

for $y \geq 0$.

The distance from a point x_0 on the x axis to a point (x,y) above the x axis can be found by integration. If we consider a point directly above x_0 then $dx = 0$ and we have

$$dS^2 = \frac{dy^2}{y^2} \tag{2}$$

Integrating, we have

$$S = \int_0^{y_0} \frac{dy}{y} = \ln y\Big|_0^{y_0} = \ln y_0 + \infty = \infty \tag{3}$$

Thus the distance from any point on the x axis to a point in the upper half plane is infinite.

The geodesic equations can be found from the Lagrangian, which can be found from 1. We have

$$dS = \int_0^1 d\sigma \sqrt{\frac{1}{y^2}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]} \tag{4}$$

where σ is a parameter which varies from 0 to 1 along the desired path. The Lagrangian is then

$$L = \sqrt{\frac{1}{y^2}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]} \tag{5}$$

Lagrange's equations are

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(dx^i/d\sigma\right)}\right) = \frac{\partial L}{\partial x^i} \tag{6}$$

where $x^1 = x$ and $x^2 = y$. From 5 we have

$$\begin{aligned}
\frac{\partial L}{\partial (dx/d\sigma)} &= \frac{1}{Ly^2}\frac{dx}{d\sigma} \\
\frac{\partial L}{\partial (dy/d\sigma)} &= \frac{1}{Ly^2}\frac{dy}{d\sigma} \\
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial y} &= -\frac{1}{Ly^3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right]
\end{aligned} \tag{7}$$

The geodesic equations are thus

$$\frac{d}{d\sigma}\left(\frac{1}{Ly^2}\frac{dx}{d\sigma}\right) = 0 \tag{8}$$

$$\frac{d}{d\sigma}\left(\frac{1}{Ly^2}\frac{dy}{d\sigma}\right) = -\frac{1}{Ly^3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] \tag{9}$$

From 4, we have

$$L = \frac{dS}{d\sigma} \tag{10}$$

We can write 8 as

$$\frac{d}{d\sigma}\left(\frac{1}{Ly^2}\frac{dx}{d\sigma}\right) = \frac{d}{d\sigma}\left(\frac{1}{y^2}\frac{dx}{d\sigma}\frac{d\sigma}{dS}\right) = \frac{d}{d\sigma}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 \tag{11}$$

Multiplying by $d\sigma/dS$ we have

$$\frac{d}{dS}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 \tag{12}$$

The RHS of 9 can be multiplied by $d\sigma/dS$ to get

$$-\frac{1}{Ly^3}\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] = -\frac{1}{y^3}\left(\frac{d\sigma}{dS}\right)^2\left[\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2\right] \tag{13}$$

$$= -\frac{1}{y^3}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] \tag{14}$$

The LHS of 9 can be transformed in the same way as 8 to give

$$\frac{d}{dS}\left(\frac{1}{y^2}\frac{dy}{dS}\right) = -\frac{1}{y^3}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] \tag{15}$$

Thus in terms of arc length, the geodesic equations are 12 and 15:

$$\boxed{\begin{aligned} &\frac{d}{dS}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 \\ &\frac{d}{dS}\left(\frac{1}{y^2}\frac{dy}{dS}\right) = -\frac{1}{y^3}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] \end{aligned}} \tag{16}$$

From 12 we integrate to get

$$\frac{1}{y^2}\frac{dx}{dS} = a \equiv \text{constant} \tag{17}$$

so that

$$\frac{dx}{dS} = ay^2 \tag{18}$$

If $a = 0$, then $dx/dS = 0$, which means that x doesn't change along the corresponding geodesic. Thus one geodesic is a vertical straight line.

If $a \neq 0$, then we have from dividing both sides of 1 by dS^2:

$$\frac{1}{y^2}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] = 1 \tag{19}$$

Inserting 18 we have

$$\frac{dy}{dS} = \sqrt{y^2 - a^2y^4} \tag{20}$$

Dividing 18 by 20 we have

$$\frac{dx}{dy} = \frac{ay^2}{\sqrt{y^2 - a^2y^4}} = \frac{ay}{\sqrt{1 - a^2y^2}} \tag{21}$$

This can be integrated to give

$$\int dx = \int \frac{ay\,dy}{\sqrt{1 - a^2y^2}} \tag{22}$$

$$x = x_0 - \frac{1}{a}\sqrt{1 - a^2y^2} \tag{23}$$

where x_0 is a constant of integration. Rearranging and squaring both sides gives

$$(x-x_0)^2 = \frac{1}{a^2}\left(1-a^2y^2\right) = \frac{1}{a^2} - y^2 \tag{24}$$

or

$$(x-x_0)^2 + y^2 = \frac{1}{a^2} \tag{25}$$

This is the equation of a semicircle (since we're requiring $y \geq 0$) with centre on the x axis at $x = x_0$ and radius $1/a$. Thus a geodesic is either a vertical line or a semicircle.

We can integrate 18 and 20 to find x and y as functions of the arc length S. From 20 we have (using Maple for the integral)

$$S = \int \frac{dy}{y\sqrt{1-a^2y^2}} = -\tanh^{-1}\left[\frac{1}{\sqrt{1-a^2y^2}}\right] \tag{26}$$

This can be inverted to give

$$y = \frac{\sqrt{\tanh^2 S - 1}}{a \tanh S} \tag{27}$$

Inserting this into 18 we have

$$x = \frac{1}{a \tanh S} + x_0 \tag{28}$$

From 27 and 28 we see that 25 is satisfied. However, there is a problem in that the tanh function takes on values in the interval $(-1, 1)$ whenever its argument is real. Thus 27 implies that y is an imaginary number. The other possibility is that S is a complex number, but this doesn't seem to make much sense, since it's an arc length. I haven't been able to find a resolution to this problem, so any comments are welcome.