INVARIANT HYPERBOLAS

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We've seen that the space-time interval between two events, defined as

$$\Delta s^2 \equiv -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{1}$$

is an invariant. That is, the interval between two events is the same as measured by all inertial observers. Returning to one spatial dimension, we can write the equation on which all pairs of events with a given interval lie:

$$(\Delta x)^2 - (\Delta t)^2 = -a^2 \tag{2}$$

for some constant a. Since $-a^2$ is always negative (we'll consider $+a^2$ below), such intervals can have $\Delta x = 0$ but not $\Delta t = 0$. That is, such events are always separated by a non-zero time interval, but it is possible for them to be observed to occur at the same place (although not all observers will see these two events in the same place). This type of interval is *timelike*.

If we take one of these events to be the origin, then we can write this equation in the form

$$x^2 - t^2 = -a^2 \tag{3}$$

which is the equation of a hyperbola. In Fig. 1, we've plotted the upper branch of two such hyperbolas, one (in magenta) with $a = 1$ and the other (in dark blue) with $a = 2$. For the magenta hyperbola, the event E represents the point at which observer O_1 (in the lab frame, with coordinate axes of x_1 and t_1) sees $x_1 = 0$, and the point F is where O_2 (in the moving frame, with coordinate axes x_2 (red) and t_2 (light blue) sees $x_2 = 0$. Thus the two observers disagree about events at which one of the observers (O_1 here) sees the two events occurring at the same place. (There's nothing mysterious about this, since Galilean relativity predicts the same thing. A person in a moving train sees himself as at the same place in his frame, while an observer standing beside the track and watching the train go by sees the man moving.)

Similarly for the dark blue hyperbola; here O_1 sees event A occurring at $x_1 = 0$ but O_2 says it is event B that occurs at $x_2 = 0$.

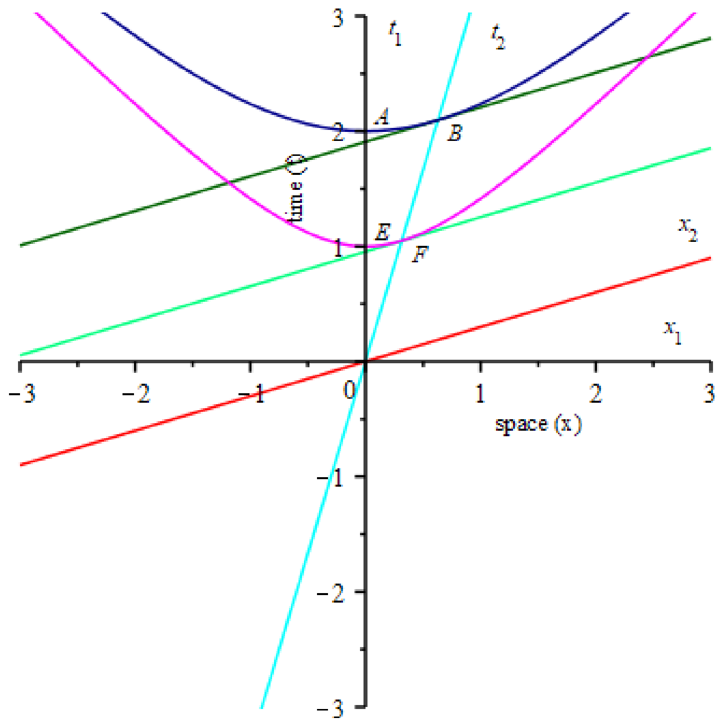

FIGURE 1. Invariant hyperbolas.

Also, we can use these hyperbolas to calibrate one axis with respect to the other. In the diagram, if we use O_1's axes as the starting point, we see that O_1 measures event E (on the magenta hyperbola) to occur at $x_1 = 0$, $t_1 = 1$ while O_2 measures event F to occur at $x_2 = 0$, $t_2 = 1$. Thus as the hyperbola intersects the time axes of the various observers, it marks off the points at which each observer measures his local time variable to be 1. Similarly, the dark blue hyperbola marks off all the points where each observer measures his local time as 2.

One question is whether these calibration points are linear. That is, if we choose the hyperbolas which intersect the t_1 axis at values $t_1 = 1$ and $t_1 = a$, is the ratio of the two times equal to a in all other frames as well? For example, in the figure, we know that the distances along the t_1 axis obey $0E = EA = 1$, but is it true that $0F = FB$ as well? Clearly the distances $0F$ and FB as measured on the space-time diagram are not equal to 1, but are they equal to each other so that the ratio of time measurements in O_2's system is the same as that in O_1's system?

We can verify this with a bit of algebra. We saw in an earlier post that the t_2 axis has a slope of $1/v$, where v is the relative speed of the two systems.

Therefore we can find the intersection point between the hyperbola 3 and the t_2 axis with equation (in O_1's frame)

$$t = \frac{1}{v}x \tag{4}$$

Substituting the line's equation into 3, we get

$$x^2\left(1 - \frac{1}{v^2}\right) = -a^2 \tag{5}$$

$$x = \frac{av}{\sqrt{1-v^2}} \tag{6}$$

$$t = \frac{a}{\sqrt{1-v^2}} \tag{7}$$

This is the intersection point of the t_2 axis and the hyperbola (e.g. the point B if $a = 2$, or F if $a = 1$). The distance from the origin to this point, as measured by O_1, is

$$d = \sqrt{x^2 + t^2} \tag{8}$$

$$= \sqrt{\frac{1+v^2}{1-v^2}}a \tag{9}$$

That is, the distance from the origin to the intersection point is directly proportional to a, so the ratios of time intervals in various frames are indeed the same.

Another feature of the invariant hyperbola that is useful is that the tangent line at the intersection point between the hyperbola and time axis is always parallel to the space axis in any inertial frame. This can be seen in the diagram where the tangent to the lower hyperbola at point F is drawn in light green and can be seen to be parallel to the x_2 axis, drawn in red. Similarly, the tangent to the upper hyperbola at point B (drawn in dark green) is also parallel to the x_2 axis.

Although it seems to be true from the graph, this can also be shown algebraically. Using implicit differentiation with respect to x on 3, we get

$$2x - 2t\frac{dt}{dx} = 0 \tag{10}$$

$$\frac{dt}{dx} = \frac{x}{t} \tag{11}$$

We can now plug in the values of x and t at the intersection point from 6 and 7 to get

$$\left.\frac{dt}{dx}\right|_{t_2\text{ axis}} = \frac{av}{\sqrt{1-v^2}}\frac{\sqrt{1-v^2}}{a} \tag{12}$$

$$= v \tag{13}$$

We saw in an earlier post that the slope of the x_2 axis is indeed v so the tangent lines are parallel to the x_2 axis.

Looking specifically at the magenta hyperbola (with $a = 1$), we see that event F has time coordinate $t_2 = 1$ in O_2's system, but as measured in O_1's system, the time as calculated from 7 when we calculated the coordinates of the intersection point is $t_1 = 1/\sqrt{1-v^2}$. That is, O_1 thinks more time has elapsed between the event at the origin and the event at F than O_2 does, so O_1 thinks O_2's clock is running slowly. This is the time dilation effect.

Time dilation works in the opposite direction as well. To see this, we can refer to Fig. 1 again. The time between events 0 and E, as measured by O_1 is $t_1 = 1$. To find the corresponding t_2 for event E, we need a line parallel to the x_2 axis that intersects event E. We can see from Fig. 1 that this line is parallel to the light green line and lies slightly above it, so we know that the time t_2 as measured by O_2 will be larger than the time t_1 as measured by O_1. Thus to O_2, O_1's clock appears to be running slow.

To work out the actual quantity, we can use a bit of algebra. We'll do the calculations in O_1's frame for simplicity. We know that, to O_1, this line has slope v and t_1-intercept $t_1 = 1$, so its equation is

$$t = vx + 1 \tag{14}$$

The equation of the t_2 axis is

$$t = \frac{x}{v} \tag{15}$$

To find where these two lines intersect, we set $x = vt$ in 14 and solve for t to obtain

$$\left(1-v^2\right)t = 1 \tag{16}$$

or, using

$$\gamma \equiv \frac{1}{\sqrt{1-v^2}} \tag{17}$$

we have

$$t = \gamma^2 \tag{18}$$

This is the time (as measured by O_1) for an event on the t_2 axis that occurs at the same time (to O_2) as the event E. To convert to O_2's frame, we use

7 which tells us that a time interval of $t_1 = \gamma$ between two events on the t_2 axis as measured by O_1 corresponds to $t_2 = 1$ as measured by O_2, so we divide 18 by γ to get the time of event E as measured by O_2:

$$t_2(E) = \frac{\gamma^2}{\gamma} = \gamma \tag{19}$$

Thus a time interval of $t_1 = 1$ between two events in O_1's rest frame is measured by O_2 to be a time interval of γ, so that O_2 experiences exactly the same time dilation factor as O_1.

We can do a similar analysis with hyperbolas that have the form

$$x^2 - t^2 = +a^2 \tag{20}$$

These curves allow $t = 0$ but not $x = 0$ and are known as *spacelike* curves. Thus events separated by a positive interval can be seen by some observers to be simultaneous, but never at the same place. Using similar analyses to what we did here for timelike curves, these hyperbolas can be used to calibrate the x axes of various observers, and we find that the ratios of distances are the same to all observers. Finally, the tangent line to the intersection of the hyperbola with the x axis is parallel to the t axis.