KILLING VECTORS

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A general metric in general relativity is written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor and the dx^μ are differentials of the coordinates. In general, the components of $g_{\mu\nu}$ depend on the coordinates. However, if $g_{\mu\nu}$ is independent of a particular coordinate, say x^1, then the distance element ds^2 is invariant under a change of that coordinate. This follows because if we increment x^1 by some constant k so that $x^1 \rightarrow x^1 + k$, then dx^1 does not change.

A vector that lies along a direction in which the metric doesn't change is known as a *Killing vector*, named after the German mathematician Wilhelm Killing. In the above example, a Killing vector would be

$$\xi^\alpha = (0,1,0,0) \tag{2}$$

where the four coordinates are x^μ with $\mu = 0,1,2,3$ as usual.

A Killing vector need not be a unit vector, so any multiple of 2 would also qualify.

In general, the metric of some complex system in general relativity will depend on some or all of the coordinates, so Killing vectors may be difficult or impossible to find. For flat space, however, things get considerably simpler. In rectangular coordinates, the flat space metric in three dimensions is

$$dS^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

This metric is independent of all three coordinates (x,y,z), so we can list three Killing vectors as

$$\boldsymbol{\xi} = \begin{cases} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{cases} \tag{4}$$

We can also write the flat space 3-d metric in spherical coordinates as

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{5}$$

In this case, the metric *does* depend on two of the coordinates: r and θ, so we can't associate a Killing vector with either of these coordinates. However, the metric is independent of ϕ, so in spherical coordinates, the Killing vector in the ϕ direction is

$$\xi^\alpha = (0, 0, 1) \tag{6}$$

where the coordinates are listed in the order (r, θ, ϕ). This means that we can change ϕ by adding a constant to it, without changing the metric. This is equivalent to a rotation in the xy plane about the z axis.

This vector can be written in rectangular coordinates. In the xy plane (or any plane with constant θ), the basis vectors are

$$\boldsymbol{e}_r = (r\cos\theta, r\sin\theta, 0) \tag{7}$$

$$\boldsymbol{e}_\phi = (-r\sin\theta, r\cos\theta, 0) \tag{8}$$

The form for $\boldsymbol{e}_\phi$ follows from the requirement that it is perpendicular to $\boldsymbol{e}_r$. In rectangular coordinates we have $x = r\cos\theta$ and $y = r\sin\theta$, so these vectors are equivalent to

$$\begin{aligned} \boldsymbol{e}_{r,z} &= (x, y, 0) \\ \boldsymbol{e}_{\phi,z} &= (-y, x, 0) \end{aligned} \tag{9}$$

where the z in the subscript is a reminder that we're rotating about the z axis. Thus the vector $\boldsymbol{e}_{\phi,z}$ is another Killing vector.

There are two other Killing vectors corresponding to rotations about the x and y axes. These can be obtained by permuting the labels above. For example, if we consider a rotation about the x axis, then the rotation occurs in the yz plane. The angle θ is now measured relative to the x axis (rather than the z axis, as above), and ϕ is the rotation in the yz plane. We thus permute $x \to y$ and $y \to z$ to get the new Killing vector. That is, in $\boldsymbol{e}_{\phi,z}$ in 9, the x component becomes y. Since it appears in the y (second) slot in 9, it is displaced to the z slot. Similarly, the $-y$ in 9 becomes $-z$ and is displaced from the x (first) slot to the y slot. Thus the Killing vector becomes

$$\boldsymbol{e}_{\phi,x} = (0, -z, y) \tag{10}$$

We can do a similar exercise for rotations about the y axis. In this case, we permute $x \to z$ and $y \to x$, so we have

$$\boldsymbol{e}_{\phi,y} = (z, 0, -x) \tag{11}$$

In both cases, we could have taken the opposite direction for the basis vector, so we could also have

$$\begin{aligned} \boldsymbol{e}_{\phi,x} &= (0, z, -y) \\ \boldsymbol{e}_{\phi,y} &= (-z, 0, x) \end{aligned} \tag{12}$$

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