LENGTH CONTRACTION

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In addition to time dilation, the other main kinematic prediction of special relativity is *length contraction*. The formula can be derived using a similar "light bulb in a train" experiment that we used to derive time dilation. As usual, we have a train moving at speed v along a straight track, and an observer T on the train and a second observer G on the ground. We have a light bulb at one end of the train's car and a mirror at the other end. If the length of the car, as measured by T, is Δx_T, then the time taken for the light to make a round trip from the bulb to the mirror and back is (again, using $c=1$)

$$\Delta t_T = 2\Delta x_T \tag{1}$$

To G, however, the the outward and return trips of the light take different times, due to the motion of the train. The outward journey must cover a distance of Δx_G (the length of the train's car as measured by G) plus the distance the train travels in time it takes the light to reach the other end of the car. This travel distance is $v\Delta t_{Go}$, where Δt_{Go} is the time measured by G for the light to travel the length of the car. Thus we have

$$\Delta t_{Go} = \Delta x_G + v\Delta t_{Go} \tag{2}$$

On the return journey, the distance travelled by the light is again the length Δx_G of the car, but now we subtract the distance travelled by the train since the train is moving in the opposite direction to the light. This gives us for the return time Δt_{Gr} as measured by G:

$$\Delta t_{Gr} = \Delta x_G - v\Delta t_{Gr} \tag{3}$$

Note that we are assuming that the length of the car may be different to the two observers. We can solve these two equations for the times and get

$$\Delta t_{Go} = \frac{\Delta x_G}{1-v} \tag{4}$$

$$\Delta t_{Gr} = \frac{\Delta x_G}{1+v} \tag{5}$$

The total time measured by G is therefore

$$\begin{aligned}\Delta t_G &= \Delta t_{Go} + \Delta t_{Gr} && (6)\\ &= \Delta x_G \left(\frac{1}{1-v} + \frac{1}{1+v}\right) && (7)\\ &= \Delta x_G \left(\frac{2}{1-v^2}\right) && (8)\\ &= 2\Delta x_G \gamma^2 && (9)\end{aligned}$$

The two times are related by the time dilation formula. To apply this correctly, we need to note that T uses only the one clock to measure both the departure and arrival of the light, since these two events happen at the same place in his frame. Observer G must use two clocks, since the train moves relative to G between the events. Thus T's time is the proper time and must be less than G's time, so

$$\Delta t_G = \gamma \Delta t_T \tag{10}$$

Combining this with 1 and 9 we get

$$\begin{aligned}2\Delta x_G \gamma^2 &= 2\gamma \Delta x_T && (11)\\ \Delta x_G &= \frac{\Delta x_T}{\gamma} && (12)\end{aligned}$$

Therefore the length of the car as measured by G is shorter than the length measured by T, by the factor γ. This is the length (or Lorentz) contraction effect.

We can also derive length contraction from Lorentz transformations. The transformations here are (since T is moving to the right with speed v relative to G):

$$\begin{aligned}\Delta t_G &= \gamma(\Delta t_T + v\Delta x_T)\\ \Delta x_G &= \gamma(\Delta x_T + v\Delta t_T)\end{aligned} \tag{13}$$

The key point here is that, to measure the length of the car, G must observe both ends of the car *at the same time* in G's frame. That is, we must have $\Delta t_G = 0$. From 13, we have

$$\Delta t_G = 0 \tag{14}$$

which implies

$$\Delta t_T = -v\Delta x_T \tag{15}$$

We can now plug this into the equation for Δx_G and we find

$$\Delta x_G = \gamma\left(\Delta x_T - v^2\Delta x_T\right) \tag{16}$$
$$= \frac{1-v^2}{\sqrt{1-v^2}}\Delta x_T \tag{17}$$
$$= \frac{\Delta x_T}{\gamma} \tag{18}$$

as before.

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