LINEARITY OF LORENTZ TRANSFORMATIONS

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In special relativity, we usually have two inertial frames with parallel rectangular axes and with one frame S' moving at speed v along the x axis relative to the 'lab' frame S which is taken to be stationary relative to the observer. When the Lorentz transformations giving the relations between the spacetime coordinates (x,t) in S to the coordinates (x',t') in S' are derived, it is often just assumed that the transformations must be linear. In fact, we can derive this condition from the postulates of relativity.

For simplicity, we'll use the standard configuration described above with S' moving at speed v along the x axis of frame S, and we'll take the origins of the two frames to be located at the same spacetime point when $t = t' = 0$. Suppose that the relations between the two sets of coordinates are given by some general functions, so we have

$$\begin{aligned} x &= f\left(x',t'\right) \\ t &= g\left(x',t'\right) \end{aligned} \tag{1}$$

We now take differentials of these equations:

$$dx = \frac{\partial f}{\partial x'}dx' + \frac{\partial f}{\partial t'}dt' \tag{2}$$

$$dt = \frac{\partial g}{\partial x'}dx' + \frac{\partial g}{\partial t'}dt' \tag{3}$$

Dividing the first equation by the second, we get

$$\frac{dx}{dt} = \frac{\frac{\partial f}{\partial x'}dx' + \frac{\partial f}{\partial t'}dt'}{\frac{\partial g}{\partial x'}dx' + \frac{\partial g}{\partial t'}dt'} \tag{4}$$

$$= \frac{\frac{\partial f}{\partial x'}\frac{dx'}{dt'} + \frac{\partial f}{\partial t'}}{\frac{\partial g}{\partial x'}\frac{dx'}{dt'} + \frac{\partial g}{\partial t'}} \tag{5}$$

From the postulates of relativity, the laws of physics must look the same in all inertial frames, which means that if an object is moving at a constant velocity in one frame, it must be moving at a constant (but not necessarily the same) velocity in another inertial frame. The quantity $\frac{dx}{dt}$ is the velocity of an object in frame S, so if this is constant, then its velocity as measured in frame S', which is $\frac{dx'}{dt'}$, must also be constant. That means that all the partial derivatives of f and g on the RHS of 5 must be constants. If they weren't, then $\frac{dx}{dt}$ would depend on x' and/or t', meaning that the velocity of the object would depend on its spacetime location, which is false.

If all the partial derivatives are constants, then the functions f and g must have the form

$$f\left(x',t'\right) = Ax' + Bt' + C \tag{6}$$

$$g\left(x',t'\right) = Dx' + Et' + F \tag{7}$$

Since we're taking the origins of the two frames to coincide (that is, $(x,t) = (0,0)$ corresponds to $(x',t') = (0,0)$) we must have $C = F = 0$, so the transformations must have form

$$x = Ax' + Bt' \tag{8}$$

$$t = Dx' + Et' \tag{9}$$

From here, we can go on to derive the Lorentz transformations.

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