

LOCAL FLAT FRAME FOR A CIRCULAR ORBIT

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Here's another example of calculating a flat local reference frame in a global Schwarzschild metric. We'll follow the same procedure as in the case of the observer falling radially inwards. This time, we're interested in an observer in a circular orbit of radius r . First, we need to work out the four-velocity in Schwarzschild coordinates. We can get this from the conserved quantities e and ℓ .

$$r^2 \frac{d\phi}{d\tau} = \ell \tag{1}$$

$$\left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} = e \tag{2}$$

For a circular orbit, we can get ℓ and e using our earlier formulas:

$$\ell^2 = \frac{r^2 GM}{r - 3GM} \tag{3}$$

$$e = \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{1/2} \tag{4}$$

Therefore

$$\frac{d\phi}{d\tau} = \frac{\ell}{r^2} \tag{5}$$

$$= \frac{1}{r} \sqrt{\frac{GM}{r - 3GM}} \tag{6}$$

$$\frac{dt}{d\tau} = \frac{e}{1 - 2GM/r} \tag{7}$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{1/2} \tag{8}$$

Since there is no motion in the radial or θ directions, the four-velocity is

$$\mathbf{o}_t = \mathbf{u} = \left[\left(1 - \frac{2GM}{r}\right)^{-1} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{1/2}, 0, 0, \frac{1}{r} \sqrt{\frac{GM}{r-3GM}} \right] \quad (9)$$

This can be simplified a bit by using some algebra. The first component is

$$\left(1 - \frac{2GM}{r}\right)^{-1} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right) \right]^{1/2} = \quad (10)$$

$$\left[\frac{r - 3GM - GM + 4(GM)^2/r}{(r - 3GM)(1 - 2GM/r)^2} \right]^{1/2} = \quad (11)$$

$$\left[\frac{r \left(1 - 4GM/r + 4(GM/r)^2\right)}{(r - 3GM)(1 - 2GM/r)^2} \right]^{1/2} = \quad (12)$$

$$\left[\frac{r(1 - 2GM/r)^2}{(r - 3GM)(1 - 2GM/r)^2} \right]^{1/2} = \frac{1}{\sqrt{1 - \frac{3GM}{r}}} \quad (13)$$

We can therefore rewrite 9 as

$$\mathbf{o}_t = \mathbf{u} = \frac{1}{\sqrt{1 - \frac{3GM}{r}}} \left[1, 0, 0, \sqrt{\frac{GM}{r^3}} \right] \quad (14)$$

As a check, we can verify that $\mathbf{u} \cdot \mathbf{u} = -1$ (this is most easily done with Maple, but if you're persistent it can be done by hand).

Now we align the x , y and z local directions with the global ϕ , $-\theta$ and r directions respectively. For \mathbf{o}_y and \mathbf{o}_z , this means they will have the same global components as for a stationary observer, so that

$$\mathbf{o}_y = \left[0, 0, -\frac{1}{r}, 0 \right] \quad (15)$$

$$\mathbf{o}_z = \left[0, \sqrt{1 - \frac{2GM}{r}}, 0, 0 \right] \quad (16)$$

To find \mathbf{o}_x , we start with $\mathbf{o}_x \cdot \mathbf{o}_t = 0$, which gives us

$$g_{tt}o_t^t o_x^t + g_{\phi\phi}o_t^\phi o_x^\phi = 0 \quad (17)$$

$$-\left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{1/2} o_x^t + r \sqrt{\frac{GM}{r-3GM}} o_x^\phi = 0 \quad (18)$$

$$\left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{-1/2} r \sqrt{\frac{GM}{r-3GM}} o_x^\phi = o_x^t \quad (19)$$

Now we can use normalization, so that $\mathbf{o}_x \cdot \mathbf{o}_x = 1$. We get

$$\left\{1 - \left(1 - \frac{2GM}{r}\right) \frac{GM}{r-3GM} \left[1 - \frac{GM}{r} \left(1 - \frac{3GM}{r}\right)^{-1} \left(1 - \frac{4GM}{r}\right)\right]^{-1}\right\} r^2 (o_x^\phi)^2 = 1 \quad (20)$$

The algebra is tedious so is best handled with Maple, and after simplifying, we get

$$o_x^\phi = \frac{1}{r} \sqrt{\frac{r-2GM}{r-3GM}} \quad (21)$$

We can substitute this back into the earlier equation to get o_x^t , and, after simplifying, we get

$$\mathbf{o}_x = \left[\sqrt{\frac{rGM}{(r-2GM)(r-3GM)}}, 0, 0, \frac{1}{r} \sqrt{\frac{r-2GM}{r-3GM}} \right] \quad (22)$$

The formula breaks down for $r \leq 3GM$, which is presumably because if a photon approaches more closely than that, it spirals into $r = 0$. In particular, for $r \leq 3GM$, it is not possible for dr/dt to be zero, which we assumed in this derivation.