LOCAL FLATNESS THEOREM

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General relativity postulates that spacetime is composed of a manifold characterized by a metric tensor. A key property of a manifold is that, at each point, it is *locally flat*, that is, it can be approximated by the flat spacetime used in special relativity. The flat spacetime metric is

$$\left[\eta_{\alpha\beta}\right]=\left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{1}$$

The local flatness theorem says that, at any point $\mathcal{P}$, the metric $g_{\alpha\beta}$ for points x^{μ} near $\mathcal{P}$ has the form

$$g_{\alpha\beta}\left(x^{\mu}\right)=\eta_{\alpha\beta}+\mathcal{O}\left(\left(x^{\mu}-\mathcal{P}^{\mu}\right)^{2}\right) \tag{2}$$

That is, the space is flat up to first order in the coordinate differences $(x^{\mu}-\mathcal{P}^{\mu})$.

To prove the theorem, we must show that there is a coordinate transformation from some arbitrary system x^{α} to a special system $x^{\alpha'}$, where the primed system reduces to an inertial frame at the point $\mathcal{P}$. That is

$$x^{\alpha}=x^{\alpha}\left(x^{\mu'}\right) \tag{3}$$

$$\Lambda^{\alpha}{}_{\mu'}=\frac{\partial x^{\alpha}}{\partial x^{\mu'}} \tag{4}$$

where $\Lambda^{\alpha}{}_{\mu'}$ is the Lorentz transformation between the two systems. The metric therefore transforms as

$$g_{\mu'\nu'}=\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}g_{\alpha\beta} \tag{5}$$

The goal, therefore, is to find a coordinate system $x^{\mu'}$ where $g_{\mu'\nu'}$satisfies 2 to first order.

To do this, we expand $\Lambda^{\alpha}{}_{\mu'}$ in a Taylor series at a point $\vec{x}$ near the point $\mathcal{P}$. We'll denote values at $\mathcal{P}$ by a subscript 0, as we're effectively taking $\mathcal{P}$ to be the origin. From 4 we have

$$\Lambda^{\alpha}{}_{\mu'}(\vec{x}) = \Lambda^{\alpha}{}_{\mu'}(\mathcal{P}) + \left(x^{\gamma'} - x_0^{\gamma'}\right) \frac{\partial \Lambda^{\alpha}{}_{\mu'}}{\partial x^{\gamma'}}(\mathcal{P}) + \frac{1}{2}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\lambda'} - x_0^{\lambda'}\right) \frac{\partial^2 \Lambda^{\alpha}{}_{\mu'}}{\partial x^{\gamma'} \partial x^{\lambda'}}(\mathcal{P}) + \ldots \quad (6)$$

From 4, $\Lambda^{\alpha}{}_{\mu'}$ is a first derivative, so the derivatives in 6 involve second and third derivatives of $x^{\alpha}\left(x^{\mu'}\right)$. That is, we have

$$\Lambda^{\alpha}{}_{\mu'}(\vec{x}) = \Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} + \left(x^{\gamma'} - x_0^{\gamma'}\right) \frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'} \partial x^{\mu'}}\Big|_{\mathcal{P}} + \frac{1}{2}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\lambda'} - x_0^{\lambda'}\right) \frac{\partial^3 x^{\alpha}}{\partial x^{\gamma'} \partial x^{\lambda'} \partial x^{\mu'}}\Big|_{\mathcal{P}} + \ldots \quad (7)$$

We now Taylor expand the metric 5. From the product rule, there will be three terms in the expansion that are first order in $\left(x^{\gamma'} - x_0^{\gamma'}\right)$. We get

$$\begin{aligned} g_{\mu'\nu'}(\vec{x}) = {} & \Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} \Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} + \\ & \left(x^{\gamma'} - x_0^{\gamma'}\right)\Big[\Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} \Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta,\gamma'}\big|_{\mathcal{P}} + \\ & \Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} \frac{\partial^2 x^{\beta}}{\partial x^{\gamma'} \partial x^{\nu'}}\Big|_{\mathcal{P}} + \\ & \Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} \frac{\partial^2 x^{\alpha}}{\partial x^{\gamma'} \partial x^{\mu'}}\Big|_{\mathcal{P}}\Big] + \\ & \frac{1}{2}\left(x^{\gamma'} - x_0^{\gamma'}\right)\left(x^{\lambda'} - x_0^{\lambda'}\right)[\ldots] \end{aligned} \quad (8)$$

where the $[\ldots]$ in the last line contains terms with third derivatives of x^{α} and second derivatives $g_{\alpha\beta,\gamma'\lambda'}$ of the metric.

We now need to see if we can find a coordinate system such that 8 can be made to satisfy 2. First, we need to clarify what we know before we start. It's assumed that we are given the metric $g_{\alpha\beta}$ and thus know it and all its derivatives. What we don't know, but wish to determine, are the $\Lambda^{\alpha}{}_{\mu'}$ matrices and its derivatives. How many free variables do we have?

The matrix $\Lambda^{\alpha}{}_{\mu'}$ has $4 \times 4 = 16$ components, and all of these are, in principle, independent, since we don't know the form of the transformation 4. The first condition to be satisfied is that the metric $g_{\mu'\nu'}(\mathcal{P}) = \eta_{\mu'\nu'}$, that is, that the spacetime is locally flat at the point $\mathcal{P}$. Since the metric $g_{\mu'\nu'}$ is symmetric, it has 10 independent components, so from 8, we wish to satisfy

$$g_{\mu'\nu'}(\mathcal{P}) = \Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} \Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} = \eta_{\mu'\nu'} \tag{9}$$

This is a set of 10 equations and we have 16 free variables in the form of the $\Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}}$ matrix. Thus we have enough free variables to satisfy the equations, with 6 left over. These extra 6 variables correspond to the degrees of freedom in a Lorentz transformation. A Lorentz boost has 3 components (the direction of the boost), and a rotation also has 3 components.

Now what about the first order terms in 8? The second derivatives $\frac{\partial^2 x^\beta}{\partial x^{\gamma'}\partial x^{\nu'}}$ are symmetric in γ' and ν' since it doesn't matter which order we take the derivatives, so for each value of β there are $\binom{4}{2} = 6$ ways of choosing γ' and ν' so that they are different, and 4 ways of choosing them so that they are the same, so for each value of β there are $6+4 = 10$ independent derivatives, giving a total of $4\times 10 = 40$ in all.

The first order terms in 8 contain the derivatives $g_{\alpha\beta,\gamma'}\big|_{\mathcal{P}}$ of which there are $10\times 4 = 40$ as well. Thus the first order condition becomes

$$\begin{aligned}
&\Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} \Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta,\gamma'}\big|_{\mathcal{P}} + \\
&\Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} \frac{\partial^2 x^\beta}{\partial x^{\gamma'}\partial x^{\nu'}}\Big|_{\mathcal{P}} + \\
&\Lambda^{\beta}{}_{\nu'}\Big|_{\mathcal{P}} g_{\alpha\beta}\big|_{\mathcal{P}} \frac{\partial^2 x^\alpha}{\partial x^{\gamma'}\partial x^{\mu'}}\Big|_{\mathcal{P}} = 0
\end{aligned} \tag{10}$$

which is a system of 40 equations, one for each component of $g_{\alpha\beta,\gamma'}\big|_{\mathcal{P}}$.

At this stage, we have already determined the values of $\Lambda^{\alpha}{}_{\mu'}\Big|_{\mathcal{P}}$ to satisfy 9, so the only free variables we have are the 40 second derivatives $\frac{\partial^2 x^\beta}{\partial x^{\gamma'}\partial x^{\nu'}}$. Since we have 40 free variables for 40 equations, this condition can also be satisfied, which verifies that the local flatness theorem 2 is true.

We can go one step further and investigate whether it's possible to make the second order term zero as well. The second order term involves the second derivatives of the metric $g_{\alpha\beta,\gamma'\lambda'}$. There are 10 independent components $g_{\alpha\beta}$ of the metric due to its symmetry. The two derivatives indexed by γ' and λ' also have 10 independent values (since the order of the derivatives doesn't matter, as mentioned above), so there are $10\times 10 = 100$ independent components of $g_{\alpha\beta,\gamma'\lambda'}$. Thus to have the second order terms equal to zero, we need to satisfy 100 simultaneous equations. For this purpose, we have the third derivatives $\frac{\partial^3 x^\alpha}{\partial x^{\gamma'}\partial x^{\lambda'}\partial x^{\mu'}}\Big|_{\mathcal{P}}$. How many of these are there?

Again, we note that the order of the derivatives doesn't matter. There are $\binom{4}{3}=4$ ways of choosing γ', λ' and μ' all different, and 4 ways of choosing them so they are all the same. If two of the indexes are the same and the other is different, there are $2\binom{4}{2}=12$ ways of choosing them. This is because there are $\binom{4}{2}$ ways of choosing 2 items from 4, but for each such choice, two of γ', λ' and μ'could have one value and the other one have the other value, or vice versa. For example, we could have $\gamma'=\lambda'=1$ and $\mu'=2$ or $\gamma'=\lambda'=2$ and $\mu'=1$.

Thus for each α there are $\binom{4}{3}+2\binom{4}{2}+4=20$ independent components, so in total there are $4\times20=80$ independent components. That is, we have only 80 variables in 100 simultaneous equations, so we don't have enough variables to satisfy the condition 10. Thus it's possible to obtain a locally flat metric up to first order, but not up to second order. It is the second order terms which give rise to the notion of the curvature of spacetime that is central to general relativity.

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