LORENTZ INVARIANCE OF ELECTRIC CHARGE

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Post date: 19 April 2022.

In this post, we'll look at some of the implications of electric charge in special relativity.

First, consider the usual two frames S and S' in special relativity. We have a cubic box of side length L and volume L^3 at rest in the S frame. The box contains a uniform charge density ρ_0. Since the box is at rest in S, its four-current density is

$$j^{\mu}=\left(c\rho_{0},\vec{0}\right) \tag{1}$$

where $\vec{0}$ is the space vector $\vec{0}\equiv(0,0,0)$.

Frame S' is boosted in the $+x$ direction with speed v. Thus the box appears Lorentz contracted to S', reducing its volume to L^3/γ where

$$\gamma=\frac{1}{\sqrt{1-v^{2}/c^{2}}} \tag{2}$$

If we assume that the total charge in the box is seen to be the same to both observers, the charge density in S' is therefore

$$\rho'=\gamma\rho_{0} \tag{3}$$

Because the box appears to be moving in the $-x'$ direction in S', the current density is

$$\vec{j}=\left(-v\rho',0,0\right) \tag{4}$$

and its four-current is

$$j'^{\mu}=\left(c\rho',-v\rho',0,0\right) \tag{5}$$
$$=\left(c\gamma\rho_{0},-v\gamma\rho_{0},0,0\right) \tag{6}$$

We can verify that j'^{μ} are j^{μ} are 4-vectors by noting that one can be obtained from the other by a Lorentz transformation. We have

$$j'^0 = \gamma\left(j^0 - vj^1/c\right) \tag{7}$$
$$= c\gamma\rho_0 \tag{8}$$
$$j'^1 = \gamma\left(j^1 - vj^0/c\right) \tag{9}$$
$$= -\gamma v\rho_0 \tag{10}$$

Now we'd like to prove that the total charge Q in a system is indeed constant in all inertial frames. In frame S, the total charge is

$$Q = \int d^3x\ \rho\left(t = 0, \vec{x}\right) \tag{11}$$

In S', we can write the same integral in terms of the primed coordinates as

$$Q' = \int d^3x'\ \rho'\left(t' = 0, \vec{x}'\right) \tag{12}$$

We'd like to prove that $Q = Q'$.

The problem is that the measurement of time differs in the two frames. In particular, the events that occur at $t = 0$ in S, but at different spatial locations, do *not* all occur at the same time t' in S'.

To make progress, we'll assume that we can define a four-current j^μ in any inertial frame, and that the charge conservation relation holds, that is

$$\partial_\mu j^\mu = 0 \tag{13}$$

Using Lorentz transformations, we can write $\rho'\left(t' = 0, \vec{x}'\right)$ in terms of the unprimed variables. We have

$$c\rho'\left(t' = 0, \vec{x}'\right) = \gamma\left[c\rho\left(t = 0, \vec{x}\right) - \frac{v}{c}j^x\left(t = 0, \vec{x}\right)\right] \tag{14}$$

We'd like to write the arguments on the RHS in terms of the primed coordinates, so we use the inverse Lorentz transformation to get

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{15}$$
$$x = \gamma\left(x' + vt'\right) \tag{16}$$

Using $t' = 0$ and substituting into 14 we have

$$c\rho'\left(t' = 0, \vec{x}'\right) = \gamma\left[c\rho\left(\frac{\gamma v x'}{c^2}, \gamma x', y', z'\right) - \frac{v}{c}j^x\left(\frac{\gamma v x'}{c^2}, \gamma x', y', z'\right)\right] \tag{17}$$

So we can write Q' as

$$Q' = \int d^3x'\, \gamma \left[\rho \left(\frac{\gamma v x'}{c^2}, \gamma x', y', z' \right) - \frac{v}{c^2} j^x \left(\frac{\gamma v x'}{c^2}, \gamma x', y', z' \right) \right] \tag{18}$$

We can now change variables in the integral according to:

$$\begin{aligned} x &= \gamma x' \\ y &= y' \\ z &= z' \\ d^3x' &= \frac{d^3x}{\gamma} \end{aligned} \tag{19}$$

We get

$$Q' = \int d^3x \left[\rho \left(\frac{vx}{c^2}, x, y, z \right) - \frac{v}{c^2} j^x \left(\frac{vx}{c^2}, x, y, z \right) \right] \tag{20}$$

As a check, at $v = 0$, we have

$$Q' = \int d^3x \rho (t = 0, x, y, z) = Q \tag{21}$$

We would like to show that $\frac{dQ'}{dv} = 0$ indicating that Q' is independent of the relative speed v of S and S'. This would prove that the total charge is indeed seen to be the same in all inertial frames. From 20 we have

$$\frac{dQ'}{dv} = \int d^3x \left[\frac{x}{c^2} \frac{\partial \rho}{\partial t} - \frac{1}{c^2} j^x - \frac{v}{c^2} \frac{x}{c^2} \frac{\partial j^x}{\partial t} \right] \tag{22}$$

$$= \frac{1}{c^2} \int d^3x \left[x \left(\frac{\partial \rho}{\partial t} - \frac{v}{c^2} \frac{\partial j^x}{\partial t} \right) - j^x \right] \tag{23}$$

We now invoke the continuity condition 13. This gives, since $j^y = j^z = 0$:

$$\frac{\partial j^0}{\partial x^0} = \frac{\partial (c\rho)}{\partial (ct)} = \frac{\partial \rho}{\partial t} = -\frac{\partial j^x}{\partial x} \tag{24}$$

We now have

$$\frac{dQ'}{dv} = \frac{1}{c^2} \int d^3x \left[-x \left(\frac{\partial j^x}{\partial x} \left(\frac{vx}{c^2}, x, y, z \right) + \frac{v}{c^2} \frac{\partial j^x}{\partial t} \right) - j^x \right]_{t = \frac{vx}{c^2}, x, y, z} \tag{25}$$

Here the notation on the RHS indicates that the time is constrained to be $\frac{vx}{c^2}$ while the three spatial coordinates are all independent. It's important to note here that the first term in parentheses is the derivative of j^x with respect to x where the argument contains a time that depends on x according to

$$t = \frac{vx}{c^2} \tag{26}$$

By using the chain rule for a function of several variables, we have

$$\frac{\partial j^x}{\partial x}\left(\frac{vx}{c^2}, x, y, z\right) = \left[\frac{v}{c^2}\frac{\partial j^x}{\partial t} + \frac{\partial j^x}{\partial x}\right]\Bigg|_{t=\frac{vx}{c^2},x,y,z} \tag{27}$$

Substituting this into 25, we have

$$\frac{dQ'}{dv} = \frac{1}{c^2}\int d^3x\,\left[-x\frac{\partial j^x}{\partial x} - j^x\right]_{t=\frac{vx}{c^2},x,y,z} \tag{28}$$

We can now integrate the first term in the integrand by parts. The derivative $\frac{\partial j^x}{\partial x}$ is the equivalent of a divergence, since $j^y = j^z = 0$, so the integral of this term becomes

$$-\frac{1}{c^2}\int d^3x\left(x\nabla\cdot\vec{j}\right) \tag{29}$$

When we integrate the divergence over a volume, we can use Gauss's theorem to convert it to a surface integral. As usual, we assume that all currents go to zero sufficiently fast as we approach infinity, so the integral is zero. Completing the integration by parts thus leads to

$$-\frac{1}{c^2}\int d^3x\left(x\nabla\cdot\vec{j}\right) = \frac{1}{c^2}\int d^3x\left(\frac{\partial x}{\partial x}j^x\right) \tag{30}$$

$$= \frac{1}{c^2}\int d^3x\; j^x \tag{31}$$

Inserting this back into 28 we see that the j^x terms cancel and we're left with

$$\frac{dQ'}{dv} = 0 \tag{32}$$

as required. QED.