LORENTZ TRANSFORMATIONS

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Post date: 15 Jan 2021.

The Lorentz transformations allow conversions between the coordinates of two observers O_1 and O_2, where O_2 moves with speed v along the x_1 axis of O_1.

We saw in an earlier post that coordinates perpendicular to the direction of motion are the same in both coordinate systems, and we also know that the transformation between the systems is linear, the most general transformation is

$$t_2 = at_1 + bx_1 \tag{1}$$
$$x_2 = dt_1 + ex_1 \tag{2}$$
$$y_2 = y_1 \tag{3}$$
$$z_2 = z_1 \tag{4}$$

where the constant coefficients a, b, d and e are functions of v (since v is the relative velocity of the two frames and this is a constant, the constants a, b, d and e could depend on v).

We can use a space-time diagram (Fig. 1) to work out these coefficients. In the diagram, the main axes are those of $O_1$with the horizontal axis being the spatial dimension x and the vertical axis being the time dimension t. As usual, we're taking $c = 1$.

The axes of O_2 are drawn in green (time t_2 axis) and red (space x_2 axis).

We've seen that the slope of the t_2 axis is $1/v$ and of the x_2 axis is v. So when $x_2 = 0$, we must have for the t_2 axis (from 2):

$$dt_1 + ex_1 = 0 \tag{5}$$

The coordinates (t_1, x_1) of a point on the t_2 axis must satisfy the slope condition, so we have

$$t_1 = \frac{1}{v} x_1 \tag{6}$$

Combining with 5, we have

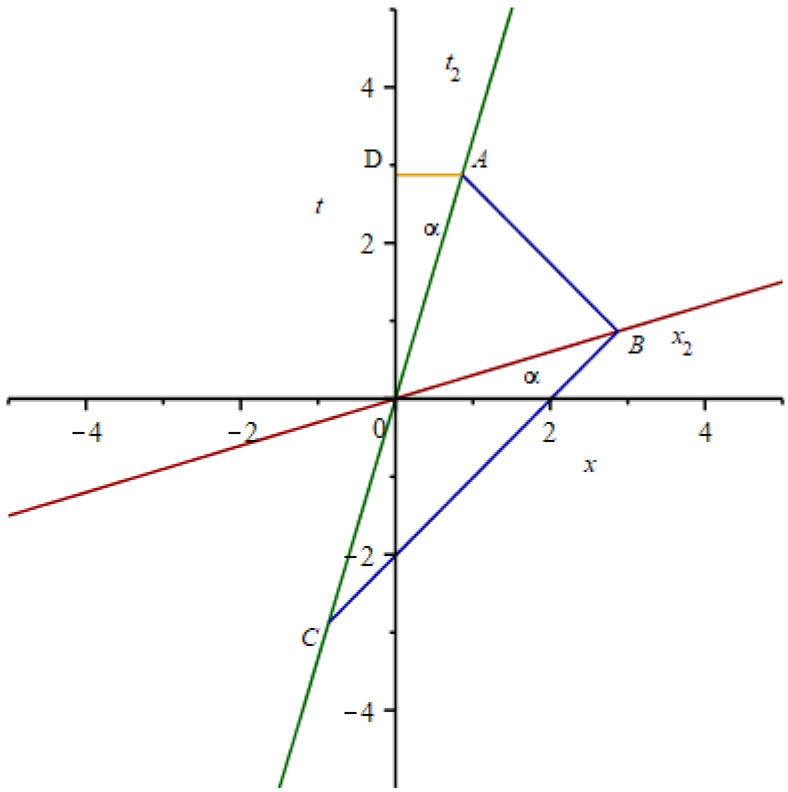

FIGURE 1. Lorentz transformations.

$$t_1 = -\frac{e}{d}x_1 = \frac{1}{v}x_1 \tag{7}$$

which gives a relation between d and e:

$$\frac{d}{e} = -v \tag{8}$$

Similarly, when $t_2 = 0$, we must have for the x_2 axis (from 1):

$$at_1 + bx_1 = 0 \tag{9}$$

The coordinates (t_1, x_1) of a point on the x_2 axis must satisfy the slope condition, so we have

$$t_1 = vx_1 \tag{10}$$

Combining with 9 we have

$$t_1 = -\frac{b}{a}x_1 = vx_1 \tag{11}$$

so we have

$$\frac{b}{a} = -v \tag{12}$$

We can now write the transformations in terms of two coefficients.

$$t_2 = a(t_1 - vx_1) \tag{13}$$
$$x_2 = e(x_1 - vt_1) \tag{14}$$

From Fig. 1, for the x_2 axis, if we pick $x_2 = 1$, then the corresponding (x_1, t_1) coordinates are the sides of a right-angled triangle, so

$$\begin{aligned} x_1 &= \cos\alpha \\ t_1 &= \sin\alpha \end{aligned} \tag{15}$$

so from 14:

$$1 = e(\cos\alpha - v\sin\alpha) \tag{16}$$

Similarly, for the t_2 axis, if we pick $t_2 = 1$, then the corresponding (x_1, t_1) coordinates are

$$\begin{aligned} t_1 &= \cos\alpha \\ x_1 &= \sin\alpha \end{aligned} \tag{17}$$

so from 13

$$1 = a(\cos\alpha - v\sin\alpha) \tag{18}$$

Comparing 16 and 18 shows that

$$a = e \tag{19}$$

so we have

$$\begin{aligned} t_2 &= a(t_1 - vx_1) \\ x_2 &= a(x_1 - vt_1) \end{aligned} \tag{20}$$

These are the transformations from a frame O_1 to another frame O_2 that is moving with speed $+v$ along their common x axes. If we start in O_2's frame, however, and want to transform back to O_1, the transformations should be the same but with v replaced by $-v$, since O_1 is moving to the left with speed $-v$ relative to O_2. That is, we must have

$$\begin{aligned} t_1 &= a(t_2 + vx_2) \\ x_1 &= a(x_2 + vt_2) \end{aligned} \tag{21}$$

We can use 20 and 21 to determine the constant a. We substitute 20 into 21 to get

$$t_1 = a^2\left(t_1 - vx_1 + vx_1 - v^2 t_1\right) \tag{22}$$

$$= a^2 t_1\left(1 - v^2\right) \tag{23}$$

from which we get

$$a = \pm\frac{1}{\sqrt{1 - v^2}} \tag{24}$$

We must take the plus sign for a so that 20 and 21 give $t_1 = t_2$ and $x_1 = x_2$ when $v = 0$.

This gives us the final form for the Lorentz transformations

$$\begin{aligned} t_2 &= \frac{1}{\sqrt{1 - v^2}}(t_1 - vx_1) \\ x_2 &= \frac{1}{\sqrt{1 - v^2}}(x_1 - vt_1) \\ y_2 &= y_1 \\ z_2 &= z_1 \end{aligned} \tag{25}$$

As the factor $\frac{1}{\sqrt{1-v^2}}$ is very common in relativity, it's often given a symbol of its own:

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2}} \tag{26}$$

so the transformations are often written as

$$\begin{aligned} t_2 &= \gamma(t_1 - vx_1) \\ x_2 &= \gamma(x_1 - vt_1) \\ y_2 &= y_1 \\ z_2 &= z_1 \end{aligned} \tag{27}$$

These transformations apply to the special case of inertial frames that are aligned so that O_2 travels with speed v along O_1's x_1 axis, with the corresponding y and z axes parallel in the two systems. This includes all real-life situations, although in some cases the coordinate systems may be more complex and the transformations take on a more involved form. However, all the standard relativistic effects such as time dilation and length contraction can be derived from these Lorentz transformations.

Since the Lorentz transformations are linear in the spacetime coordinates, they apply also to spacetime *intervals*, as well as absolute coordinates. That is, if we have an interval $(\Delta t_1, \Delta x_1)$ in frame O_1 we can find the corresponding interval in frame O_2:

$$\begin{aligned} \Delta t_2 &= \gamma(\Delta t_1 - v\Delta x_1) \\ \Delta x_2 &= \gamma(\Delta x_1 - v\Delta t_1) \end{aligned} \tag{28}$$

In this form, we can verify the invariance of the spacetime interval Δs^2. We have

$$\Delta s_2^2 = -\Delta t_2^2 + \Delta x_2^2 \tag{29}$$

$$= \gamma^2 \left[-(\Delta t_1 - v\Delta x_1)^2 + (\Delta x_1 - v\Delta t_1)^2 \right] \tag{30}$$

$$= \gamma^2 \left[-\Delta t_1^2 \left(1 - v^2\right) + \Delta x_1^2 \left(1 - v^2\right) + 2v\Delta t_1 \Delta x_1 - 2v\Delta t_1 \Delta x_1 \right] \tag{31}$$

$$= \gamma^2 \left(1 - v^2\right) \left(-\Delta t_1^2 + \Delta x_1^2\right) \tag{32}$$

$$= -\Delta t_1^2 + \Delta x_1^2 \tag{33}$$

$$= \Delta s_1^2 \tag{34}$$

where we used 26 in the fourth line.

We can also see from 28 that two observers may not agree on the order (in time) of two events. If $\Delta t_1 > 0$, for example, then Δt_2 can be positive, negative or zero, depending on the $v\Delta x_1$ term. Thus observer O_2 may think that the two events occurred in the opposite order to that measured by O_1. If $\Delta t_2 = 0$, then O_2 thinks the events are simultaneous, but O_1 will disagree.

This shouldn't be a cause for concern for us in our everyday lives, since these effects occur only if v is significantly larger than the infinitesimal values it normally has in our surroundings. For very small v, $\Delta t_2 \approx \Delta t_1$.

Comments

From Kevin, 4 May 2021 22:58

To get equations 15 and 17, aren't you having to assume a Euclidean geometry? I didn't think the corresponding coordinates (x1, t1) could be found by treating the triangle formed by the hypotenuse (x2) as we would in a typical euclidean space. Wouldn't there have to be some other constant in front of the sin and cos? If so, I suppose the constant would be irrelevant in showing a=e. Am I thinking about that correctly?

Reply: I think that spacetime diagrams are drawn on Euclidean axes, which is why curves of constant spacetime separation are hyperbolas on such diagrams, rather than straight lines. That is, a curve of constant spacetime separation is given by $\Delta s^2 = -\Delta t^2 + \Delta x^2$, which is a hyperbola for a given value of Δs. So it's correct to use things like Pythagoras's theorem in such diagrams.

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