MANIFOLDS, CURVES AND SURFACES

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Relativity calculations usually take place on *manifolds*, which are geometric objects that consist of a collection of points, each point satisfying the condition that the area of the manifold near that point resembles Euclidean (that is, 'flat') space. This definition isn't very precise, but for the purposes of relativity, it is probably enough to understand that the general idea is that, although the overall structure of space-time is *not* Euclidean, locally it *seems* Euclidean, which is why Newtonian physics (which assumes Euclidean geometry) works so well on small distance scales and in regions of weak gravitational fields.

The number *n* of dimensions needed to define a manifold is the number of parameters needed to locate any point on it. For example, a straight line is a one dimensional manifold, since it can be defined using a single parameter t by the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{p}_0 \tag{1}$$

Here, $\mathbf{r}_0 = (r_{0x}, r_{0y})$ is one point on the line, and $\mathbf{p}_0 = (p_{0x}, p_{0y})$ is a vector parallel to the line. The parameter t runs from $-\infty$ to $+\infty$ and in the process, traces out the entire line.

In terms of rectangular coordinates, this equation can be written in parametric form:

$$x = r_{0x} + p_{0x}t \tag{2}$$

$$y = r_{0y} + p_{0y}t \tag{3}$$

We can eliminate t to get a *constraint form* of the equation:

$$t = \frac{1}{p_{0x}}(x - r_{0x}) \tag{4}$$

$$y = r_{0y} + \frac{p_{0y}}{p_{0x}}(x - r_{0x}) \tag{5}$$

$$= \frac{p_{0y}}{p_{0x}}x + \left(r_{0y} - \frac{p_{0y}}{p_{0x}}r_{0x}\right) \tag{6}$$

The last equation is the more familiar slope-intercept form of a line: $y = mx + b$. In this case, the entire manifold is Euclidean, since a straight line is a 'flat' one-dimensional space. You might argue that we need more than one parameter to define a line, since we had to specify $\boldsymbol{r}_0$ and $\boldsymbol{p}_0$ as well as t. However, $\boldsymbol{r}_0$ and $\boldsymbol{p}_0$ are constants for any given line. It is only the parameter t that needs to vary to trace out the line, so in that sense, a line is a one-dimensional manifold.

A circle is also a one-dimensional manifold, since again, it needs only a single parameter to trace out the curve. We can write a circle of radius a centred at the origin in parametric form:

$$x = a\cos\theta \tag{7}$$

$$y = a\sin\theta \tag{8}$$

where the parameter θ runs from 0 to 2π. The parameter can be eliminated by squaring each equation and adding to get the constraint form:

$$x^2 + y^2 - a^2 = 0 \tag{9}$$

In this case, the circle itself is non-Euclidean, but if we look at a small enough part of it, it 'resembles' a patch of a one-dimensional Euclidean space (that is, a straight line segment).

A sphere of radius a centred at the origin is a two-dimensional manifold, since it requires two parameters to locate a point on the sphere, and if we restrict ourselves to a small enough patch on the sphere, it resembles a portion of a plane, which is a two-dimensional Euclidean space. The parametric equations of the sphere are:

$$x = a\sin\theta\cos\phi \tag{10}$$

$$y = a\sin\theta\sin\phi \tag{11}$$

$$z = a\cos\theta \tag{12}$$

Here, θ is the usual angle between the z axis and the radius vector, and ϕ is the angle between the x axis and the projection of the radius vector into the xy plane. These two parameters can be eliminated by squaring and adding all three equations to get the constraint

$$x^2 + y^2 + z^2 - a^2 = 0 \tag{13}$$

More generally, we can consider a manifold of any number n of dimensions and look at curves and surfaces within that manifold. For an n-dimensional manifold, we need n coordinates to specify any given point.

For reasons that will become apparent later (that is, in a future post), tensor theory writes these coordinates with a superscript, so they are denoted $x^1, x^2 \ldots x^{n-1} x^n$. Obviously, there is the potential of confusing these superscripts with exponents, so we need to be careful with the notation. If we want to write an exponent, it's usual to enclose the coordinate in parentheses, so that $\left(x^1\right)^2$ is the square of coordinate x^1.

To specify a curve or surface within the manifold, we need to know the dimension of the curve or surface. (A curve is always defined as one-dimensional, but the term 'surface' can have any number of dimensions from 2 up to n.) For a subsurface (where the number of dimensions m is strictly less than n) we need m parameters u^i to define it, so we get

$$x^a = x^a\left(u^1, u^2, \ldots, u^m\right) \tag{14}$$

where this represents n equations, one for each $a \in \{1, 2, \ldots, n\}$. This is the generalization of the examples above, in which we embedded a one-dimensional subsurface (the circle) in a two-dimensional manifold (a plane), and a two-dimensional subsurface (the sphere) in a three-dimensional manifold (Euclidean 3-d space).

If we have m dimensions in the subsurface embedded in an n-dimensional manifold, we must have $n-m$ constraints which can be written in the form

$$f^b\left(x^1, x^2, \ldots, x^n\right) = 0; \quad b = 1 \ldots m-n \tag{15}$$

We saw the constraint equations in the examples above. In those examples, the dimension of the subsurface was always one less than the dimension of the space in which they were embedded, so we needed only one constraint. These constraint equations can be obtained by eliminating the parameters u^i from the parametric equations, as we did above.

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