MERCATOR MAP METRIC

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For the Mercator projection of the map of the world, the grid coordinates (x,y) on the 2-d map are given in terms of the usual spherical coordinates (θ,ϕ) by

$$x = \frac{W}{2\pi}\phi \tag{1}$$

$$y = -\frac{W}{2\pi}\ln\tan\left(\frac{\theta}{2}\right) \tag{2}$$

where W is a constant. Note that θ is not the same as latitude (which is 0 at the equator and $\pm\frac{\pi}{2}$ at the poles), but is the usual spherical angle which is 0 at the north pole, $\frac{\pi}{2}$ at the equator and π at the south pole.

We can relate the usual spherical metric, which is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{3}$$

to the Mercator metric. We'll start by working out dx^2+dy^2. I've used Maple to do some simplifications.

$$dx^2+dy^2 = \frac{W^2}{4\pi^2}\left[d\phi^2 + \frac{1}{4}\frac{\left(1+\tan^2\frac{\theta}{2}\right)^2}{\tan^2\frac{\theta}{2}}\right]d\theta^2 \tag{4}$$

To simplify this, we can use the trig identity

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \tag{5}$$

Using this, we get, after a few trig identities

$$dx^2+dy^2 = \frac{W^2}{4\pi^2}\left[\frac{d\theta^2}{\sin^2\theta} + d\phi^2\right] \tag{6}$$

$$= \frac{W^2}{4\pi^2\sin^2\theta}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{7}$$

Comparing this to 3, we see that

$$ds^2 = \Omega^2\left(dx^2+dy^2\right) \tag{8}$$

with

$$\Omega = \frac{2\pi \sin\theta}{W} \tag{9}$$

Note that $dx^2 + dy^2$ blows up to infinity at the poles, where $\theta = 0$ or π. This is the cause of the distances at high latitudes (near the poles) appearing much greater than the corresponding distances near the equator, with the result that Greenland appears to be the same size as South America, even though in reality it's only about one-eighth as big.

We can transform this by writing 2 in the form

$$\tan\frac{\theta}{2} = e^{-2\pi y/W} \tag{10}$$

and another trig identity

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \tag{11}$$

Using these, we have (with $s \equiv \sin\theta$, $c \equiv \cos\theta$ and $e \equiv e^{-2\pi y/W}$):

$$s - e = ce \tag{12}$$

$$(s-e)^2 = \left(1 - s^2\right) e^2 \tag{13}$$

$$s^2\left(1+e^2\right) - 2se = 0 \tag{14}$$

Discounting the $s = \sin\theta = 0$ solution, we have, restoring the original variables

$$\sin\theta = \frac{2e^{-2\pi y/W}}{1+e^{-4\pi y/W}} \tag{15}$$

$$= \frac{2}{e^{2\pi y/W} + e^{-2\pi y/W}} \tag{16}$$

$$= \frac{1}{\cosh\frac{2\pi y}{W}} \tag{17}$$

Plugging this back into 9 we have

$$\Omega = \frac{2\pi}{W\cosh\frac{2\pi y}{W}} \tag{18}$$