The inverse metric tensor $g^{ij}$ is defined so that

$$g^{ij} g_{jk} = \delta^i_k$$  \hspace{1cm} (1)

If the metric tensor is viewed as a matrix, then this is equivalent to saying $[g^{ij}] = [g_{ij}]^{-1}$. The transformation property of $g^{ij}$ can be worked out by direct calculation, using the transformation of $g_{ij}$ and the fact that $\delta^i_k$ is invariant.

$$g^{ij} g'_{jk} = \delta^i_k$$  \hspace{1cm} (2)

$$= g^{ij} \frac{\partial x^l}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} g_{lm}$$  \hspace{1cm} (3)

We can try the transformation

$$g^{ij} = \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^j}{\partial x^b} g^{ab}$$  \hspace{1cm} (4)

Substituting, we get

$$g^{ij} g'_{jk} = \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^j}{\partial x^b} g^{ab} \frac{\partial x^l}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} g_{lm}$$  \hspace{1cm} (5)

$$= \frac{\partial x'^i}{\partial x^a} g^{ab} \delta^l_b \frac{\partial x^m}{\partial x'^k} g_{lm}$$  \hspace{1cm} (6)

$$= \frac{\partial x'^i}{\partial x^a} g^{al} \frac{\partial x^m}{\partial x'^k} g_{lm}$$  \hspace{1cm} (7)

$$= \frac{\partial x'^i}{\partial x^a} \frac{\partial x^m}{\partial x'^k} \delta^a_m$$  \hspace{1cm} (8)

$$= \frac{\partial x'^i}{\partial x^a} \frac{\partial x^m}{\partial x'^k}$$  \hspace{1cm} (9)

$$= \delta^i_k$$  \hspace{1cm} (10)
On line 2 we used \( \frac{\partial x^i}{\partial x^b} \frac{\partial x^l}{\partial x^b} = \delta^i_l \) and on line 4 we used \( g^{al}g_{lm} = \delta^a_m \).

Thus \( g^{ij} \) is a rank-2 contravariant tensor, and is the inverse of \( g_{ij} \) which is a rank-2 covariant tensor. Since the matrix inverse is unique (basic fact from matrix algebra), we can use the standard techniques of matrix algebra to calculate the inverse.

In rectangular coordinates, \( g^{ij} = g_{ij} \) since the metric is diagonal with all diagonal elements equal to 1. In polar coordinates in 2-d,

\[
g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}
\]

so the inverse is

\[
g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}
\]

A contravariant vector \( v^i \) can be lowered (converted to a covariant vector) by multiplying by \( g_{ij} \):

\[
v_i = g_{ij} v^j
\]

The covariant vector can be converted back into a contravariant vector by raising its index:

\[
g^{ij} v_j = g^{ij} g_{jk} v^k = \delta^i_k v^k = v^i
\]

If we start with a vector \( v^i \) in rectangular coordinates, we can convert it to polar coordinates:

\[
v_r = v^x \cos \theta + v^y \sin \theta
\]
\[
v_\theta = -v^x \sin \theta + v^y \cos \theta
\]

We can lower these components by multiplying by \( g_{ij} \)

\[
v_r = v^x \cos \theta + v^y \sin \theta
\]
\[
v_\theta = r^2 \left( -v^x \sin \theta + v^y \cos \theta \right) = -rv^x \sin \theta + rv^y \cos \theta
\]
The magnitude is found by combining the covariant and contravariant vectors (which can also be viewed as a one-form operating on a vector, or vice versa):

\[ v^i v_i = v^r v_r + v^\theta v_\theta \]  
\[ = (v^x \cos \theta + v^y \sin \theta)^2 + (-v^x \sin \theta + v^y \cos \theta)^2 \]  
\[ = (v^x)^2 + (v^y)^2 \]  
(No implied sum on the RHS in line 1.)

The same result can be obtained by using the appropriate form of the metric on either the vector or one-form version. Using the vector, we have:

\[ v^2 = g_{ij} v^i v^j \]  
\[ = g_{rr} v^r v^r + g_{\theta\theta} v^\theta v^\theta \]  
\[ = (v^x \cos \theta + v^y \sin \theta)^2 + \frac{1}{r^2} \left( -r v^x \sin \theta + r v^y \cos \theta \right)^2 \]  
\[ = (v^x)^2 + (v^y)^2 \]

Using the one-form, we have

\[ v^2 = g^{ij} v_i v_j \]  
\[ = g^{rr} v_r v_r + g^{\theta\theta} v_\theta v_\theta \]  
\[ = (v^x \cos \theta + v^y \sin \theta)^2 + \frac{1}{r^2} \left( -r v^x \sin \theta + r v^y \cos \theta \right)^2 \]
\[ = (v^x)^2 + (v^y)^2 \]