METRIC TENSOR - INVERSE AND RAISING AND LOWERING INDICES

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The inverse metric tensor g^{ij} is defined so that

$$g^{ij}g_{jk}=\delta^{i}{}_{k} \tag{1}$$

If the metric tensor is viewed as a matrix, then this is equivalent to saying $\left[g^{ij}\right]=\left[g_{ij}\right]^{-1}$. The transformation property of g^{ij} can be worked out by direct calculation, using the transformation of g_{ij} and the fact that $\delta^{i}{}_{k}$ is invariant.

$$\begin{aligned}
g'^{ij}g'_{jk} &= \delta^{i}{}_{k} && (2)\\
&= g'^{ij}\frac{\partial x^{l}}{\partial x'^{j}}\frac{\partial x^{m}}{\partial x'^{k}}g_{lm} && (3)
\end{aligned}$$

We can try the transformation

$$g'^{ij}=\frac{\partial x'^{i}}{\partial x^{a}}\frac{\partial x'^{j}}{\partial x^{b}}g^{ab} \tag{4}$$

Substituting, we get

$$\begin{aligned}
g'^{ij}g'_{jk} &= \frac{\partial x'^{i}}{\partial x^{a}}\frac{\partial x'^{j}}{\partial x^{b}}g^{ab}\frac{\partial x^{l}}{\partial x'^{j}}\frac{\partial x^{m}}{\partial x'^{k}}g_{lm} && (5)\\
&= \frac{\partial x'^{i}}{\partial x^{a}}g^{ab}\delta^{l}{}_{b}\frac{\partial x^{m}}{\partial x'^{k}}g_{lm} && (6)\\
&= \frac{\partial x'^{i}}{\partial x^{a}}g^{al}\frac{\partial x^{m}}{\partial x'^{k}}g_{lm} && (7)\\
&= \frac{\partial x'^{i}}{\partial x^{a}}\frac{\partial x^{m}}{\partial x'^{k}}\delta^{a}{}_{m} && (8)\\
&= \frac{\partial x'^{i}}{\partial x^{m}}\frac{\partial x^{m}}{\partial x'^{k}} && (9)\\
&= \delta^{i}{}_{k} && (10)
\end{aligned}$$

On line 2 we used $\frac{\partial x'^j}{\partial x^b}\frac{\partial x^l}{\partial x'^j}=\frac{\partial x^l}{\partial x^b}=\delta^l{}_b$ and on line 4 we used $g^{al}g_{lm}=\delta^a{}_m$. Thus g^{ij} is a rank-2 contravariant tensor, and is the inverse of g_{ij} which is a rank-2 covariant tensor. Since the matrix inverse is unique (basic fact from matrix algebra), we can use the standard techniques of matrix algebra to calculate the inverse.

In rectangular coordinates, $g^{ij}=g_{ij}$ since the metric is diagonal with all diagonal elements equal to 1. In polar coordinates in 2-d,

$$g_{ij}=\left[\begin{array}{cc}1 & 0\\ 0 & r^2\end{array}\right] \tag{11}$$

so the inverse is

$$g^{ij}=\left[\begin{array}{cc}1 & 0\\ 0 & r^{-2}\end{array}\right] \tag{12}$$

A contravariant vector v^i can be *lowered* (converted to a covariant vector) by multiplying by g_{ij}:

$$v_i=g_{ij}v^i \tag{13}$$

The covariant vector can be converted back into a contravariant vector by *raising* its index:

$$\begin{aligned}
g^{ij}v_j &= g^{ij}g_{jk}v^k && (14)\\
&= \delta^i{}_k v^k && (15)\\
&= v^i && (16)
\end{aligned}$$

If we start with a vector v^i in rectangular coordinates, we can convert it to polar coordinates:

$$\begin{aligned}
v^r &= v^x\cos\theta+v^y\sin\theta && (17)\\
v^\theta &= -v^x\frac{\sin\theta}{r}+v^y\frac{\cos\theta}{r} && (18)
\end{aligned}$$

We can lower these components by multiplying by g_{ij}

$$\begin{aligned}
v_r &= v^x\cos\theta+v^y\sin\theta && (19)\\
v_\theta &= r^2\left(-v^x\frac{\sin\theta}{r}+v^y\frac{\cos\theta}{r}\right) && (20)\\
&= -rv^x\sin\theta+rv^y\cos\theta && (21)
\end{aligned}$$

The magnitude is found by combining the covariant and contravariant vectors (which can also be viewed as a one-form operating on a vector, or vice versa):

$$v^i v_i = v^r v_r + v^\theta v_\theta \tag{22}$$
$$= (v^x \cos\theta + v^y \sin\theta)^2 + (-v^x \sin\theta + v^y \cos\theta)^2 \tag{23}$$
$$= (v^x)^2 + (v^y)^2 \tag{24}$$

(No implied sum on the RHS in line 1.)

The same result can be obtained by using the appropriate form of the metric on either the vector or one-form version. Using the vector, we have:

$$v^2 = g_{ij} v^i v^j \tag{25}$$
$$= g_{rr} v^r v^r + g_{\theta\theta} v^\theta v^\theta \tag{26}$$
$$= (v^x \cos\theta + v^y \sin\theta)^2 + r^2 \left(-v^x \frac{\sin\theta}{r} + v^y \frac{\cos\theta}{r}\right)^2 \tag{27}$$
$$= (v^x)^2 + (v^y)^2 \tag{28}$$

Using the one-form, we have

$$v^2 = g^{ij} v_i v_j \tag{29}$$
$$= g^{rr} v_r v_r + g^{\theta\theta} v_\theta v_\theta \tag{30}$$
$$= (v^x \cos\theta + v^y \sin\theta)^2 + \frac{1}{r^2} (-r v^x \sin\theta + r v^y \cos\theta)^2 \tag{31}$$
$$= (v^x)^2 + (v^y)^2 \tag{32}$$

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