# METRIC TENSOR FOR SURFACE OF A SPHERE

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As an example of the metric tensor in a curved space, we'll use the surface of a sphere, but rather than the usual spherical coordinates we'll use a slight variation. We keep the azimuthal angle $\phi$ but use as the second coordinate the quantity $r$ which is the distance along the surface of the sphere measured from the north pole. If the radius of the sphere is $R$, then in terms of normal spherical coordinates, $r = R\theta$.

Curves of constant $\phi$ are the usual lines of longitude, while curves of constant $r$ are lines of latitude. The tangents to the two curves at a given point are always perpendicular, so the metric $g_{ij}$ will be diagonal, since the recipe for finding $g_{ij}$ is

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j \tag{1}$$

where the $\boldsymbol{e}_i$ are the basis vectors.

To find the diagonal components, consider an infinitesimal displacement $d\mathbf{s}$. We have

$$d\mathbf{s} = dr\mathbf{e}_r + d\phi\mathbf{e}_\phi \tag{2}$$

and our job is to find the two basis vectors.

The displacement along $\mathbf{e}_r$ is just $dr = Rd\theta$, so $\mathbf{e}_r$ is a unit vector. A displacement along $\mathbf{e}_\phi$ depends on the radius of the constant $r$ curve. In spherical coordinates, this is $R\sin\theta$, so in our new coordinate system we get the displacement as $R\sin\theta d\phi = R\sin\frac{r}{R}d\phi$. Therefore the magnitude of $\mathbf{e}_\phi$ is $R\sin\frac{r}{R}$. The metric tensor is thus

$$g_{ij} = \left[\begin{array}{cc} 1 & 0 \\ 0 & \left(R\sin\frac{r}{R}\right)^2 \end{array}\right] \tag{3}$$

In the usual spherical coordinates of $\theta$ and $\phi$, this is

$$d\boldsymbol{s} = \boldsymbol{e}_\theta d\theta + \boldsymbol{e}_\phi d\phi \tag{4}$$

The displacement in the $\theta$ direction is $R\,d\theta$, so now the magnitude of $\boldsymbol{e}_\theta$ is $R$. The displacement in the $\phi$ direction is the same as before, so we have

$$g_{ij} = \left[ \begin{array}{cc} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{array} \right] \tag{5}$$

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