METRIC TENSOR UNDER LORENTZ TRANSFORMATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 22 June 2021.

The invariant interval in special relativity can be written as

$$ds^2 = \eta_{ij} dx^i dx^j \tag{1}$$

where η_{ij} is the metric tensor in flat space, with components $\eta_{00} = -1$, $\eta_{ii} = +1$ for $i = 1,2,3$ and zero otherwise. Thus this relation is the same as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

Under a Lorentz transformation, we get

$$ds^2 = \eta_{ij} dx'^i dx'^j \tag{3}$$

$$= \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b dx^a dx^b \tag{4}$$

Since the interval is invariant, we get

$$\eta_{ij} \Lambda^i{}_a \Lambda^j{}_b dx^a dx^b = \eta_{ab} dx^a dx^b \tag{5}$$

$$\left(\eta_{ij} \Lambda^i{}_a \Lambda^j{}_b - \eta_{ab}\right) dx^a dx^b = 0 \tag{6}$$

Since the last equation must be true for any infintesimal interval, the quantity in parentheses must be zero, so

$$\boxed{\eta_{ab} = \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b} \tag{7}$$

That is, if we apply a Lorentz transformation (the *same* transformation!) to each index in the metric tensor, we get the same tensor back again.

We can multiply this equation by an inverse transformation to get

$$\left(\Lambda^{-1}\right)^a{}_k \eta_{ab} = \eta_{ij} \Lambda^i{}_a \Lambda^j{}_b \left(\Lambda^{-1}\right)^a{}_k \tag{8}$$

Multiplying a transformation by its inverse gives the identity matrix:

$$\Lambda^i{}_a \left(\Lambda^{-1}\right)^a{}_k = \delta^i{}_k \tag{9}$$

So we get

$$\left(\Lambda^{-1}\right)^{a}{}_{k}\eta_{ab}=\eta_{ij}\delta^{i}{}_{k}\Lambda^{j}{}_{b} \tag{10}$$

$$=\eta_{kj}\Lambda^{j}{}_{b} \tag{11}$$

Repeating the process, we get

$$\left(\Lambda^{-1}\right)^{b}{}_{l}\left(\Lambda^{-1}\right)^{a}{}_{k}\eta_{ab}=\eta_{kj}\Lambda^{j}{}_{b}\left(\Lambda^{-1}\right)^{b}{}_{l} \tag{12}$$

$$=\eta_{kj}\delta^{j}{}_{l} \tag{13}$$

$$=\eta_{kl} \tag{14}$$

Thus if we multiply the metric tensor by two *inverse* Lorentz transformations, we get the same tensor back.