MODIFIED SPHERICAL COORDINATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 14 Mar 2023.

Here are a couple of examples of spherical coordinates with different metrics than the usual 3-dim spherical coordinate system.

Example 1. We have a 3-dimensional space with the line element

$$dS^2 = \frac{dr^2}{1-2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{1}$$

Using the procedure introduced earlier we can calculate a distance and volume in this metric.

The radial distance between $r = 2M$ and $r = 3M$ is found by setting $d\theta = d\phi = 0$ and integrating over r.

$$\begin{aligned} R &= \int_{2M}^{3M} \sqrt{g_{rr}} dr && (2) \\ &= \int_{2M}^{3M} \frac{dr}{\sqrt{1-2M/r}} && (3) \\ &= M\left(\sqrt{3} + \ln\left(2+\sqrt{3}\right)\right) && (4) \end{aligned}$$

where I used Maple to do the integral.

Note that the distance is finite even though $g_{rr} = 1/(1-2M/r)$ blows up at $r = 2M$.

We can also calculate the volume of the spherical region between $r = 2M$ and $r = 3M$. This time, all three coordinates must be integrated over, and we have

$$\begin{aligned} V &= \int_{2M}^{3M} dr \int_0^{\pi} d\theta \int_0^{2\pi} \sqrt{g_{rr} g_{\theta\theta} g_{\phi\phi}} d\phi \, d\theta \, dr && (5) \\ &= \int_{2M}^{3M} dr \int_0^{\pi} d\theta \int_0^{2\pi} \frac{r^2 \sin\theta}{\sqrt{1-2M/r}} d\phi \, d\theta \, dr && (6) \\ &= M^3\left(32\sqrt{3}\pi + 10\pi \ln\left(2+\sqrt{3}\right)\right) && (7) \end{aligned}$$

Example 2. Consider a flat Euclidean space in four dimensions. The surface of a sphere in this space is given by

$$X^2+Y^2+Z^2+W^2=R^2 \tag{8}$$

for a constant R, with (X,Y,Z,W) being the four rectangular coordinates.

We can convert to a 4-dim spherical coordinate system with the transformation

$$\begin{aligned} X &= R\sin\chi\sin\theta\cos\phi \\ Y &= R\sin\chi\sin\theta\sin\phi \\ Z &= R\sin\chi\cos\theta \\ W &= R\cos\chi \end{aligned} \tag{9}$$

To verify that this transformation works, we calculate 8 directly, and use $\sin^2+\cos^2=1$ three times.

$$\begin{aligned} X^2+Y^2+Z^2+W^2 &= R^2\left[(\sin\chi\sin\theta\cos\phi)^2+(\sin\chi\sin\theta\sin\phi)^2+\right. \\ &\qquad \left.(\sin\chi\cos\theta)^2+\cos^2\chi\right] && (10) \\ &= R^2\left[(\sin\chi\sin\theta)^2+(\sin\chi\cos\theta)^2+\cos^2\chi\right] && (11) \\ &= R^2\left(\sin^2\chi+\cos^2\chi\right) && (12) \\ &= R^2 && (13) \end{aligned}$$

We can also find the metric on the surface of the sphere by calculating differentials. We have

$$\begin{aligned} dX &= R\left[\cos\chi\sin\theta\cos\phi\,d\chi+\sin\chi\cos\theta\cos\phi\,d\theta-\sin\chi\sin\theta\sin\phi\,d\phi\right] \\ dY &= R\left[\cos\chi\sin\theta\sin\phi\,d\chi+\sin\chi\cos\theta\sin\phi\,d\theta+\sin\chi\sin\theta\cos\phi\,d\phi\right] \\ dZ &= R\left[\cos\chi\cos\theta\,d\chi-\sin\chi\sin\theta\,d\theta\right] \\ dW &= -R\sin\chi\,d\chi \end{aligned} \tag{14}$$

To find the metric, we must calculate

$$dS^2=dX^2+dY^2+dZ^2+dW^2 \tag{15}$$

and find the coefficients of the various differentials. As you can see, this gives rise to quite a mess, so I did it using Maple. If you do it by hand, you'll need to use $\sin^2+\cos^2=1$ for various angles and then cancel terms. The result is

$$dS^2 = R^2 \left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right) \tag{16}$$

Thus the metric is

$$\begin{aligned} g_{\chi\chi} &= R^2 \\ g_{\theta\theta} &= R^2 \sin^2\chi \\ g_{\phi\phi} &= R^2 \sin^2\chi \sin^2\theta \end{aligned} \tag{17}$$

with all off-diagonal elements equal to zero. Note that if we take $d\chi = 0$ and $\chi = \frac{\pi}{2}$, we regain the usual 3-dimensional spherical metric.