MOMENTUM AND ENERGY

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The four-velocity in relativity is defined as the derivative of the spacetime position four-vector with respect to the proper time τ of the object:

$$\vec{U} \equiv \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right) \tag{1}$$

We've seen that this can be expressed in terms of the velocity $\boldsymbol{v}$ of the object as measured in any inertial frame by

$$\vec{U} = \gamma \left(1, v^1, v^2, v^3 \right) \tag{2}$$

In Newtonian mechanics, the momentum $\boldsymbol{p}$ of a particle of mass m is given by

$$\boldsymbol{p} = m\boldsymbol{v} \tag{3}$$

We can define a relativistic four-momentum by multiplying a particle's four-velocity by its mass:

$$\vec{p} = \gamma m \left(1, v^1, v^2, v^3 \right) \tag{4}$$

We can now try to work out a particle's kinetic energy in relativity. In Newtonian physics, if we apply a force F to a particle at rest through a distance dx, an amount of work W is done on the particle which (assuming that the force F is the only force acting on the particle) gives the particle a speed v and kinetic energy $\frac{1}{2}mv^2$. This is calculated as follows.

$$\text{kinetic energy} = W = \int F(x)\,dx \tag{5}$$
$$= \int m\frac{dv}{dt}dx \tag{6}$$
$$= m\int \frac{dv}{dx}\frac{dx}{dt}dx \tag{7}$$
$$= m\int \frac{dv}{dx}v\;dx \tag{8}$$
$$= m\int v\;dv \tag{9}$$
$$= \frac{1}{2}mv^2 \tag{10}$$

Because $p = mv$ in Newtonian physics, we can also write this integral as

$$W = \int v\;dp \tag{11}$$

At this point, we can replace the Newtonian dp by the relativistic dp by using 4. Assuming motion in the x^1 direction only, the spatial component of relativistic momentum is

$$p = p^1 = \frac{mv}{\sqrt{1-v^2}} \tag{12}$$

Plugging this into 11 we get for the relativistic kinetic energy K

$$K = m\int_0^{v_0} v\;d\left(\frac{v}{\sqrt{1-v^2}}\right) \tag{13}$$

The integral can be done by parts, but it's easier to just work out the differential and then use Maple to do the integral, so we have

$$d\left(\frac{v}{\sqrt{1-v^2}}\right) = \left[\frac{1}{\sqrt{1-v^2}} + \frac{v^2}{(1-v^2)^{3/2}}\right]dv \tag{14}$$

The integral is then

$$K = m\int_0^{v_0} v\left[\frac{1}{\sqrt{1-v^2}} + \frac{v^2}{\left(1-v^2\right)^{3/2}}\right]dv \tag{15}$$

$$= m\frac{1-\sqrt{1-v_0^2}}{\sqrt{1-v_0^2}} \tag{16}$$

$$= m\left(\gamma_0 - 1\right) \tag{17}$$

where

$$\gamma_0 = \frac{1}{\sqrt{1-v_0^2}} \tag{18}$$

Dropping the 0 subscript, we get for a particle moving with speed v:

$$K = m\left(\gamma - 1\right) \tag{19}$$

For small v, we can expand γ in a Taylor series to get

$$\gamma = \frac{1}{\sqrt{1-v}^2} = 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \mathcal{O}\left(v^6\right) \tag{20}$$

Thus if we take only the first two terms on the RHS, we see that, for small (non-relativistic) speeds, we have

$$K \approx \frac{1}{2}mv^2 \tag{21}$$

so the relativistic result does indeed reduce to the Newtonian value.

Einstein proposed that the relation 19 for the kinetic energy could be interpreted as

$$E = K + m = \gamma m \tag{22}$$

where E is now the *total* energy of the particle. That is, we interpret γm as the kinetic energy plus a rest mass energy of m, or, restoring the speed of light factor

$$\boxed{E_{\text{rest}} = mc^2} \tag{23}$$

This is the famous equation that everyone knows, even if they don't know what it means.

Returning to 4, we see that the 0 component of $\vec{p}$ is the same as the total energy γm of the particle. Thus the four-momentum can be written as

$$\vec{p} = \left(E, \boldsymbol{p}\right) = \gamma m\left(1, \boldsymbol{v}\right) \tag{24}$$

We can rearrange this equation to give a few useful results that appear frequently in relativity. First, we write $\vec{p}$ in terms of the four-velocity and then calculate its magnitude squared:

$$\vec{p} = m\vec{U} \tag{25}$$

$$\vec{p}\cdot\vec{p} = m^2\vec{U}\cdot\vec{U} \tag{26}$$

$$= -m^2 \tag{27}$$

Comparing with 24, we have

$$\vec{p}\cdot\vec{p} = -E^2 + \boldsymbol{p}^2 = -m^2 \tag{28}$$

or

$$\boxed{E^2 = m^2 + \boldsymbol{p}^2} \tag{29}$$

Although this was derived for particles with non-zero rest mass m, it is proposed that it is valid also for massless particles such as photons. In that case we have

$$\boxed{E = |\boldsymbol{p}|} \tag{30}$$

so that a photon's momentum (magnitude) is equal to its energy.

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