

MOMENTUM AND ENERGY

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The four-velocity in relativity is defined as the derivative of the spacetime position four-vector with respect to the proper time τ of the object:

$$\vec{U} \equiv \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right) \quad (1)$$

We've seen that this can be expressed in terms of the velocity \mathbf{v} of the object as measured in any inertial frame by

$$\vec{U} = \gamma (1, v^1, v^2, v^3) \quad (2)$$

In Newtonian mechanics, the momentum \mathbf{p} of a particle of mass m is given by

$$\mathbf{p} = m\mathbf{v} \quad (3)$$

We can define a relativistic four-momentum by multiplying a particle's four-velocity by its mass:

$$\vec{p} = \gamma m (1, v^1, v^2, v^3) \quad (4)$$

We can now try to work out a particle's kinetic energy in relativity. In Newtonian physics, if we apply a force F to a particle at rest through a distance dx , an amount of work W is done on the particle which (assuming that the force F is the only force acting on the particle) gives the particle a speed v and kinetic energy $\frac{1}{2}mv^2$. This is calculated as follows.

$$\text{kinetic energy} = W = \int F(x) dx \quad (5)$$

$$= \int m \frac{dv}{dt} dx \quad (6)$$

$$= m \int \frac{dv}{dx} \frac{dx}{dt} dx \quad (7)$$

$$= m \int \frac{dv}{dx} v dx \quad (8)$$

$$= m \int v dv \quad (9)$$

$$= \frac{1}{2} m v^2 \quad (10)$$

Because $p = mv$ in Newtonian physics, we can also write this integral as

$$W = \int v dp \quad (11)$$

At this point, we can replace the Newtonian dp by the relativistic dp by using 4. Assuming motion in the x^1 direction only, the spatial component of relativistic momentum is

$$p = p^1 = \frac{mv}{\sqrt{1-v^2}} \quad (12)$$

Plugging this into 11 we get for the relativistic kinetic energy K

$$K = m \int_0^{v_0} v d\left(\frac{v}{\sqrt{1-v^2}}\right) \quad (13)$$

The integral can be done by parts, but it's easier to just work out the differential and then use Maple to do the integral, so we have

$$d\left(\frac{v}{\sqrt{1-v^2}}\right) = \left[\frac{1}{\sqrt{1-v^2}} + \frac{v^2}{(1-v^2)^{3/2}} \right] dv \quad (14)$$

The integral is then

$$K = m \int_0^{v_0} v \left[\frac{1}{\sqrt{1-v^2}} + \frac{v^2}{(1-v^2)^{3/2}} \right] dv \quad (15)$$

$$= m \frac{1 - \sqrt{1-v_0^2}}{\sqrt{1-v_0^2}} \quad (16)$$

$$= m(\gamma_0 - 1) \quad (17)$$

where

$$\gamma_0 = \frac{1}{\sqrt{1-v_0^2}} \quad (18)$$

Dropping the 0 subscript, we get for a particle moving with speed v :

$$K = m(\gamma - 1) \quad (19)$$

For small v , we can expand γ in a Taylor series to get

$$\gamma = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \mathcal{O}(v^6) \quad (20)$$

Thus if we take only the first two terms on the RHS, we see that, for small (non-relativistic) speeds, we have

$$K \approx \frac{1}{2}mv^2 \quad (21)$$

so the relativistic result does indeed reduce to the Newtonian value.

Einstein proposed that the relation 19 for the kinetic energy could be interpreted as

$$E = K + m = \gamma m \quad (22)$$

where E is now the *total* energy of the particle. That is, we interpret γm as the kinetic energy plus a rest mass energy of m , or, restoring the speed of light factor

$$\boxed{E_{\text{rest}} = mc^2} \quad (23)$$

This is the famous equation that everyone knows, even if they don't know what it means.

Returning to 4, we see that the 0 component of \vec{p} is the same as the total energy γm of the particle. Thus the four-momentum can be written as

$$\vec{p} = (E, \mathbf{p}) = \gamma m (1, \mathbf{v}) \quad (24)$$

We can rearrange this equation to give a few useful results that appear frequently in relativity. First, we write \vec{p} in terms of the four-velocity and then calculate its magnitude squared:

$$\vec{p} = m\vec{U} \quad (25)$$

$$\vec{p} \cdot \vec{p} = m^2 \vec{U} \cdot \vec{U} \quad (26)$$

$$= -m^2 \quad (27)$$

Comparing with 24, we have

$$\vec{p} \cdot \vec{p} = -E^2 + \mathbf{p}^2 = -m^2 \quad (28)$$

or

$$\boxed{E^2 = m^2 + \mathbf{p}^2} \quad (29)$$

Although this was derived for particles with non-zero rest mass m , it is proposed that it is valid also for massless particles such as photons. In that case we have

$$\boxed{E = |\mathbf{p}|} \quad (30)$$

so that a photon's momentum (magnitude) is equal to its energy.

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