NONCOORDINATE BASES

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We've seen that, in 2-dimensional flat space, we can convert a vector from rectangular (unprimed) coordinates to polar (primed) coordinates by using

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta} \tag{1}$$

where

$$\Lambda^{\alpha'}{}_{\beta} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \tag{4}$$

The basis vectors in polar coordinates are

$$\begin{aligned} \vec{e}_r &= \cos\theta \vec{e}_x + \sin\theta \vec{e}_y \\ \vec{e}_\theta &= -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y \end{aligned} \tag{5}$$

These basis vectors are orthogonal, but $\vec{e}_\theta$ is not a unit vector, so these basis vectors are not orthonormal.

The metric tensor in polar coordinates is

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{6}$$

The inverse is

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \tag{7}$$

The corresponding one-form bases can be found by raising the index on the vector bases.

$$\begin{aligned}\tilde{\omega}^{r} &= \tilde{d}r = g^{r\beta}\vec{e}_{\beta} = \cos\theta\tilde{d}x + \sin\theta\tilde{d}y \\ \tilde{\omega}^{\theta} &= \tilde{d}\theta = g^{\theta\beta}\vec{e}_{\beta} = -\frac{1}{r}\sin\theta\tilde{d}x + \frac{1}{r}\cos\theta\tilde{d}y\end{aligned} \tag{8}$$

A one-form transformation can be written using the inverse of 4

$$\Lambda^{\alpha}{}_{\beta'} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \tag{9}$$

so we have

$$V_{\beta'} = \Lambda^{\alpha}{}_{\beta'}V_{\alpha} \tag{10}$$

We can define a set of orthonormal basis vectors by dividing $\vec{e}_{\theta}$ by r. Denoting these vectors by a hat over the variable, we have

$$\begin{aligned}\vec{e}_{\hat{r}} &= \cos\theta\vec{e}_{x} + \sin\theta\vec{e}_{y} \\ \vec{e}_{\hat{\theta}} &= \frac{1}{r}\hat{e}_{\theta} = -\sin\theta\vec{e}_{x} + \cos\theta\vec{e}_{y}\end{aligned} \tag{11}$$

The corresponding orthonormal one-form bases are

$$\begin{aligned}\tilde{\omega}^{\hat{r}} &= \cos\theta\tilde{d}x + \sin\theta\tilde{d}y \\ \tilde{\omega}^{\hat{\theta}} &= -\sin\theta\tilde{d}x + \cos\theta\tilde{d}y\end{aligned} \tag{12}$$

The question now is whether a transformation to or from an orthonormal basis can be done using a matrix such as 4 or 9. Considering one-forms, we therefore ask if there are coordinates (ξ,η) such that

$$\begin{aligned}\tilde{\omega}^{\hat{r}} &\equiv \tilde{d}\xi = \frac{\partial\xi}{\partial x}\tilde{d}x + \frac{\partial\xi}{\partial y}\tilde{d}y \\ \tilde{\omega}^{\hat{\theta}} &\equiv \tilde{d}\eta = \frac{\partial\eta}{\partial x}\tilde{d}x + \frac{\partial\eta}{\partial y}\tilde{d}y\end{aligned} \tag{13}$$

Comparing with 12, we see that this would require

$$\begin{aligned}\frac{\partial\xi}{\partial x} &= \frac{\partial\eta}{\partial y} = \cos\theta \\ \frac{\partial\xi}{\partial y} &= -\frac{\partial\eta}{\partial x} = \sin\theta\end{aligned} \tag{14}$$

If we could find (ξ,η) to satisfy these conditions, then the mixed second derivatives would have to be equal, since the order in which a mixed derivative is taken doesn't matter. That is, we would have

$$\frac{\partial}{\partial x}\frac{\partial \eta}{\partial y}=\frac{\partial}{\partial y}\frac{\partial \eta}{\partial x} \tag{15}$$

This would mean that

$$\frac{\partial}{\partial x}\cos\theta=-\frac{\partial}{\partial y}\sin\theta \tag{16}$$

Writing this in rectangular coordinates, we have

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}}\stackrel{?}{=}-\frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}} \tag{17}$$

We have

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}}=\frac{1}{\sqrt{x^2+y^2}}-\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} \tag{18}$$

$$-\frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}}=-\frac{1}{\sqrt{x^2+y^2}}+\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} \tag{19}$$

We can simplify this to get

$$\frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial}{\partial y}\frac{y}{\sqrt{x^2+y^2}}=\frac{1}{\sqrt{x^2+y^2}} \tag{20}$$

which is certainly not zero for any finite values of x and y. In other words, if we use 12 as the one-form bases for polar coordinates, we *cannot* find a coordinate system corresponding to this basis. Such a basis is called a *noncoordinate basis*.

Just to check that our original basis 8 is a coordinate basis, we can do the same calculation for it. That is, we want to find (ξ,η) such that (note that the indexes on $\tilde{\omega}$ no longer have hats, so the basis isn't orthonormal):

$$\begin{aligned}\tilde{\omega}^r &\equiv \tilde{d}\xi=\frac{\partial \xi}{\partial x}\tilde{d}x+\frac{\partial \xi}{\partial y}\tilde{d}y\\ \tilde{\omega}^\theta &\equiv \tilde{d}\eta=\frac{\partial \eta}{\partial x}\tilde{d}x+\frac{\partial \eta}{\partial y}\tilde{d}y\end{aligned} \tag{21}$$

This requires

$$\begin{aligned}\frac{\partial\xi}{\partial x} &= \cos\theta \\ \frac{\partial\xi}{\partial y} &= \sin\theta \\ \frac{\partial\eta}{\partial x} &= -\frac{1}{r}\sin\theta \\ \frac{\partial\eta}{\partial y} &= \frac{1}{r}\cos\theta\end{aligned} \tag{22}$$

The condition 15 is now

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial\eta}{\partial y} &= \frac{\partial}{\partial x}\left[\frac{1}{r}\cos\theta\right] \\ \frac{\partial}{\partial y}\frac{\partial\eta}{\partial x} &= -\frac{\partial}{\partial y}\left[\frac{1}{r}\sin\theta\right]\end{aligned} \tag{23}$$

We have

$$\frac{\partial}{\partial x}\left[\frac{1}{r}\cos\theta\right] = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] \tag{24}$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \tag{25}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} \tag{26}$$

$$-\frac{\partial}{\partial y}\left[\frac{1}{r}\sin\theta\right] = -\frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right] \tag{27}$$

$$= -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \tag{28}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left[\frac{1}{r}\cos\theta\right] \tag{29}$$

Thus in this case, 15 *can* be satisfied. We can find ξ and η by integrating a couple of 22. For example

$$\xi = \int \cos\theta \, dx \tag{30}$$
$$= \int \frac{x}{\sqrt{x^2+y^2}} dx \tag{31}$$
$$= \sqrt{x^2+y^2} \tag{32}$$
$$\eta = \int \frac{\cos\theta}{r} dy \tag{33}$$
$$= \int \frac{x}{x^2+y^2} dy \tag{34}$$
$$= \arctan\frac{y}{x} \tag{35}$$

These are the usual expressions for polar coordinates, so we have $\xi = r$ and $\eta = \theta$.

PINGBACKS

Pingback: Christoffel symbols in noncoordinate bases