NORMAL VECTORS AND UNIT NORMAL ONE-FORMS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 27 Jan 2021.

From basic linear algebra, we're familiar with the term 'normal vector' as a vector that is normal to the tangent to a line or surface. This definition requires a scalar product, as the usual test for one vector to be normal to another is that their scalar product is zero. However, the definition of a scalar product requires the intervention of a metric tensor such as $\eta_{\mu\nu}$ in flat spacetime, since the scalar product of two vectors $\vec{A}$ and $\vec{B}$ is given as

$$\vec{A}\cdot\vec{B}=\eta_{\mu\nu}A^{\mu}B^{\nu} \tag{1}$$

or, in component-free form, as

$$\vec{A}\cdot\vec{B}=\eta\left(\vec{A},\vec{B}\right) \tag{2}$$

where the metric tensor η is now regarded as an object with two slots that, when filled by vectors, return a number, in this case the scalar product.

With the introduction of one-forms, we have a way of defining the scalar product without the use of an auxiliary metric tensor. If the one-form corresponding to the vector $\vec{A}$ is written as $\tilde{A}$, the scalar product can be written as

$$\vec{A}\cdot\vec{B}=\tilde{A}\left(\vec{B}\right)=\tilde{B}\left(\vec{A}\right) \tag{3}$$

That is, the one-form is regarded as a rank-1 tensor which takes a single vector in its slot and returns a number, which is the scalar product.

We can now define a *normal one-form*. A one-form is normal to a surface at a point P if its value is zero when any vector tangent to the surface at P is inserted into the one-form's slot. That is, for a one-form $\tilde{N}$, we have

$$\tilde{N}\left(\vec{T}\right)=0 \tag{4}$$

for all vectors $\vec{T}$ which are tangent to the surface at P. Since $\tilde{N}\left(\vec{T}\right)$ is formally equivalent to the scalar product of the vector $\vec{N}$ corresponding to $\tilde{N}$ with the vector $\vec{T}$, this definition is essentially the same as requiring $\vec{N}\cdot\vec{T}=0$, but without the use of a metric.

A vector $\vec{N}$ is said to be a *normal vector* to a surface if its associated one-form $\tilde{N}$ satisfies 4. A normal vector or normal one-form is a *unit normal* if its magnitude is ± 1. We must allow -1 to be one of the magnitudes of a unit vector or one-form, since in the flat spacetime metric, magnitudes can be positive, negative or zero, from the formulas

$$\begin{aligned} \left|\vec{A}\right|^2 &= \eta_{\mu\nu} A^\mu A^\nu \\ \left|\tilde{A}\right|^2 &= \eta^{\mu\nu} A_\mu A_\nu \end{aligned} \tag{5}$$

with

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

If we consider a 3-d enclosed surface that separates spacetime into an inside region and an outside region, we can define an *outward* normal one-form as a normal one-form whose value is positive on vectors that point outwards from a surface. An outward normal vector is then the vector corresponding to the outward normal one-form. Because of the $\eta_{00} = \eta^{00} = -1$ component in the metric, this definition can have some counter-intuitive consequences.

To illustrate this, we consider a region in 2-d spacetime as shown in Fig. 1.

The region is the black square bounded by the lines $t = 0$, $x = 0$, $t = 1$ and $x = 1$. Consider first the bottom edge of the square, at $t = 0$. The downward pointing blue arrow is a vector $\vec{B}$ that points outwards from the surface. (The horizontal location of $\vec{B}$ along the bottom edge doesn't matter; it's just the direction of $\vec{B}$ that's important here.) In order for another vector $\vec{N}_{t=0}$ to qualify as an outward normal vector on this edge, its corresponding one-form must give a positive value when $\vec{B}$ is inserted into its slot. That is, we must have

$$\tilde{N}_{t=0}\left(\vec{B}\right) > 0 \tag{7}$$

We can assign components to $\vec{B}$:

$$\vec{B} = (-1, 0) \tag{8}$$

so that it points in the $-t$ direction. The operation of the one-form $\tilde{N}_{t=0}\left(\vec{B}\right)$ is therefore

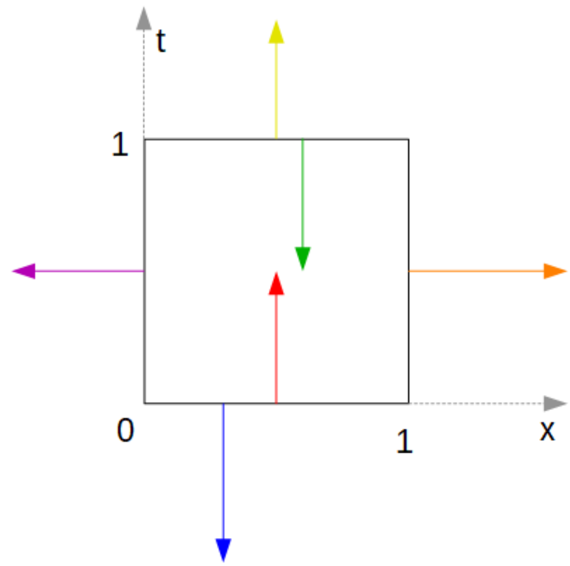

FIGURE 1. A 2-d region in spacetime.

$$\tilde{N}_{t=0}\left(\vec{B}\right) = \left[\tilde{N}_{t=0}\right]_{\mu} B^{\mu} \tag{9}$$

$$= -\left[\tilde{N}_{t=0}\right]_{0} \tag{10}$$

Therefore, if 7 is to be satisfied, we must have

$$\left[\tilde{N}_{t=0}\right]_{0} < 0 \tag{11}$$

and if we want $\tilde{N}_{t=0}$ to be a unit normal, we must have, in components:

$$\tilde{N}_{t=0} = (-1, 0) \tag{12}$$

The corresponding outward unit normal *vector* therefore has components

$$\left[\vec{N}_{t=0}\right]^{\mu} = \eta^{\mu\nu}\left[\tilde{N}_{t=0}\right]_{\nu} \tag{13}$$

so we have

$$\left[\vec{N}_{t=0}\right]^{0} = -\left[\tilde{N}_{t=0}\right]_{0} = +1 \tag{14}$$

and we find that the outward unit normal vector on the edge $t = 0$ is

$$\vec{N}_{t=0} = (+1, 0) \tag{15}$$

which is given as the red arrow in Fig. 1. That is, the *outward* unit normal vector here actually points *inward*. This counter-intuitive result is a consequence of using the Minkowski metric 6.

We find a similar result for the top edge at $t=1$. Here, we have an outward pointing vector $\vec{Y}$ is drawn in yellow and points upwards in the $+t$ direction. In components, we have

$$\vec{Y}=(+1,0) \tag{16}$$

The outward normal one-form that operates on $\vec{Y}$ must give a positive result, so we have

$$\tilde{N}_{t=1}=(+1,0) \tag{17}$$

and the corresponding outward pointing unit normal *vector* is

$$\vec{N}_{t=1}=(-1,0) \tag{18}$$

which is drawn as the downward pointing green arrow in Fig. 1. Again, because of the $\eta_{00}=-1$ element of the metric, the *outward* unit normal vector points *inward* on this edge.

For the two vertical edges, things are a bit more intuitive. For the edge $x=0$, an outward pointing vector $\vec{M}$ is drawn in magenta in Fig. 1, so we can give it components

$$\vec{M}=(0,-1) \tag{19}$$

The outward normal one-form acting on this vector must give a positive result, so we have

$$\tilde{N}_{x=0}=(0,-1) \tag{20}$$

Because the metric component $\eta_{11}=+1$, the vector $\vec{N}_{x=0}$ has the same components as the one-form:

$$\vec{N}_{x=0}=(0,-1) \tag{21}$$

so this vector is also represented by the magenta arrow in Fig. 1. In this case, the outward normal vector actually does point outwards.

A similar argument shows that on the edge $x=1$, both the outward pointing vector and the outward unit normal vector point in the same direction, drawn as the orange vector in Fig. 1.

In summary we have

Edge	$\tilde{N}$	$\vec{N}$
$t=0$	$(-1,0)$	$(1,0)$
$t=1$	$(1,0)$	$(-1,0)$
$x=0$	$(0,-1)$	$(0,-1)$
$x=1$	$(0,1)$	$(0,1)$

Given that the $t=0$ and $t=1$ edges are spacelike, this appears to be the opposite of what Schutz says at the end of his Section 3.5. The 'outward' normal vector for spacelike edges (or surfaces) points inwards, while the outward normal vector for a timelike edge points outwards, while he states that normal outward vectors on a timelike surface point inwards, and outwards on spacelike surfaces.

If a different metric is used, these conclusions are swapped. That is if we take $\eta_{00}=+1$ and $\eta_{ii}=-1$ for $i=1,2,3$, then the normal vectors would behave as Schutz states. However, he doesn't use this metric, so I'm not sure if what he says in the book is just wrong or if I've overlooked something.

Note that if we apply the one-form $\tilde{N}$ to an ordinary vector pointing outwards, we do get a positive result for all four edges. For $t=0$, for example, an ordinary outward pointing vector would be $\vec{A}=(-1,0)$, so $\tilde{N}\left(\vec{A}\right)=N_\alpha A^\alpha=(-1)\times(-1)+0\times 0=+1$. Thus it would appear that calling an ordinary (as opposed to a normal) vector 'outward' always does mean that the vector points to the outside of the surface.

Comments welcome.

Finally, suppose we add a path from $(t,x)=(0,0)$ to $(t,x)=(1,1)$, so we have a diagonal line. A vector $\vec{L}$ tangent to this line has equal components in the two directions, such as

$$\vec{L}=(1,1) \tag{22}$$

Thus the magnitude of this vector is

$$\left|\vec{L}\right|^2=\eta_{\mu\nu}L^\mu L^\nu=-1+1=0 \tag{23}$$

Thus, any one-form of the form

$$\tilde{L}=(-1,1) \tag{24}$$

will be normal to $\vec{L}$, since

$$\tilde{L}\left(\vec{L}\right)=L_\nu L^\nu=-1\times 1+1\times 1=0 \tag{25}$$

and the corresponding normal vector is just $\vec{L}$ itself. In other words, for a light-like path, the tangents and normals are parallel to each other. Since the magnitudes of both the one-forms and normal vectors are always zero, we can't create a *unit* vector or one-form for a light-like path.