NORMAL VECTORS AND UNIT NORMAL ONE-FORMS

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From basic linear algebra, we're familiar with the term 'normal vector' as a vector that is normal to the tangent to a line or surface. This definition requires a scalar product, as the usual test for one vector to be normal to another is that their scalar product is zero. However, the definition of a scalar product requires the intervention of a metric tensor such as $\eta_{\mu\nu}$ in flat spacetime, since the scalar product of two vectors $\vec{A}$ and $\vec{B}$ is given as

$$\vec{A}\cdot\vec{B}=\eta_{\mu\nu}A^{\mu}B^{\nu} \tag{1}$$

or, in component-free form, as

$$\vec{A}\cdot\vec{B}=\eta\left(\vec{A},\vec{B}\right) \tag{2}$$

where the metric tensor η is now regarded as an object with two slots that, when filled by vectors, return a number, in this case the scalar product.

With the introduction of one-forms, we have a way of defining the scalar product without the use of an auxiliary metric tensor. If the one-form corresponding to the vector $\vec{A}$ is written as $\tilde{A}$, the scalar product can be written as

$$\vec{A}\cdot\vec{B}=\tilde{A}\left(\vec{B}\right)=\tilde{B}\left(\vec{A}\right) \tag{3}$$

That is, the one-form is regarded as a rank-1 tensor which takes a single vector in its slot and returns a number, which is the scalar product.

We can now define a *normal one-form*. A one-form is normal to a surface at a point P if its value is zero when any vector tangent to the surface at P is inserted into the one-form's slot. That is, for a one-form $\tilde{N}$, we have

$$\tilde{N}\left(\vec{T}\right)=0 \tag{4}$$

for all vectors $\vec{T}$ which are tangent to the surface at P. Since $\tilde{N}\left(\vec{T}\right)$ is formally equivalent to the scalar product of the vector $\vec{N}$ corresponding to $\tilde{N}$ with the vector $\vec{T}$, this definition is essentially the same as requiring $\vec{N}\cdot\vec{T}=0$, but without the use of a metric.

A vector $\vec{N}$ is said to be a *normal vector* to a surface if its associated one-form $\tilde{N}$ satisfies 4. A normal vector or normal one-form is a *unit normal* if its magnitude is ± 1. We must allow -1 to be one of the magnitudes of a unit vector or one-form, since in the flat spacetime metric, magnitudes can be positive, negative or zero, from the formulas

$$\begin{aligned} \left|\vec{A}\right|^2 &= \eta_{\mu\nu} A^\mu A^\nu \\ \left|\tilde{A}\right|^2 &= \eta^{\mu\nu} A_\mu A_\nu \end{aligned} \tag{5}$$

with

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

If we consider a 3-d enclosed surface that separates spacetime into an inside region and an outside region, we can define an *outward* normal one-form as a normal one-form whose value is positive on vectors that point outwards from a surface. An outward normal vector is then the vector corresponding to the outward normal one-form. Because of the $\eta_{00} = \eta^{00} = -1$ component in the metric, this definition can have some counter-intuitive consequences.

To illustrate this, we consider a region in 2-d spacetime as shown in Fig. 1.

The region is the black square bounded by the lines $t = 0$, $x = 0$, $t = 1$ and $x = 1$. Consider first the bottom edge of the square, at $t = 0$. The downward pointing blue arrow is a vector $\vec{B}$ that points outwards from the surface. (The horizontal location of $\vec{B}$ along the bottom edge doesn't matter; it's just the direction of $\vec{B}$ that's important here.) In order for another vector $\vec{N}_{t=0}$ to qualify as an outward normal vector on this edge, its corresponding one-form must give a positive value when $\vec{B}$ is inserted into its slot. That is, we must have

$$\tilde{N}_{t=0}\left(\vec{B}\right) > 0 \tag{7}$$

We can assign components to $\vec{B}$:

$$\vec{B} = (-1, 0) \tag{8}$$

so that it points in the $-t$ direction. The operation of the one-form $\tilde{N}_{t=0}\left(\vec{B}\right)$ is therefore

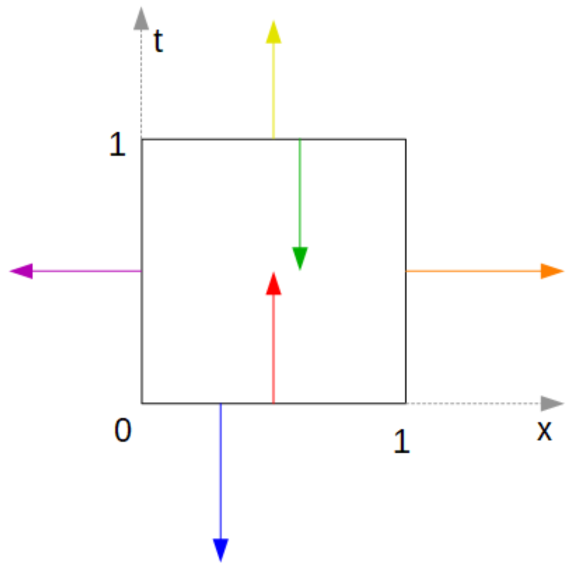

FIGURE 1. A 2-d region in spacetime.

$$\tilde{N}_{t=0}\left(\vec{B}\right) = \left[\tilde{N}_{t=0}\right]_{\mu} B^{\mu} \tag{9}$$

$$= -\left[\tilde{N}_{t=0}\right]_{0} \tag{10}$$

Therefore, if 7 is to be satisfied, we must have

$$\left[\tilde{N}_{t=0}\right]_{0} < 0 \tag{11}$$

and if we want $\tilde{N}_{t=0}$ to be a unit normal, we must have, in components:

$$\tilde{N}_{t=0} = (-1, 0) \tag{12}$$

The corresponding outward unit normal *vector* therefore has components

$$\left[\vec{N}_{t=0}\right]^{\mu} = \eta^{\mu\nu}\left[\tilde{N}_{t=0}\right]_{\nu} \tag{13}$$

so we have

$$\left[\vec{N}_{t=0}\right]^{0} = -\left[\tilde{N}_{t=0}\right]_{0} = +1 \tag{14}$$

and we find that the outward unit normal vector on the edge $t = 0$ is

$$\vec{N}_{t=0} = (+1, 0) \tag{15}$$

which is given as the red arrow in Fig. 1. That is, the *outward* unit normal vector here actually points *inward*. This counter-intuitive result is a consequence of using the Minkowski metric 6.

We find a similar result for the top edge at $t = 1$. Here, we have an outward pointing vector $\vec{Y}$ is drawn in yellow and points upwards in the $+t$ direction. In components, we have

$$\vec{Y} = (+1,0) \tag{16}$$

The outward normal one-form that operates on $\vec{Y}$ must give a positive result, so we have

$$\tilde{N}_{t=1} = (+1,0) \tag{17}$$

and the corresponding outward pointing unit normal *vector* is

$$\vec{N}_{t=1} = (-1,0) \tag{18}$$

which is drawn as the downward pointing green arrow in Fig. 1. Again, because of the $\eta_{00} = -1$ element of the metric, the *outward* unit normal vector points *inward* on this edge.

For the two vertical edges, things are a bit more intuitive. For the edge $x = 0$, an outward pointing vector $\vec{M}$ is drawn in magenta in Fig. 1, so we can give it components

$$\vec{M} = (0,-1) \tag{19}$$

The outward normal one-form acting on this vector must give a positive result, so we have

$$\tilde{N}_{x=0} = (0,-1) \tag{20}$$

Because the metric component $\eta_{11} = +1$, the vector $\vec{N}_{x=0}$ has the same components as the one-form:

$$\vec{N}_{x=0} = (0,-1) \tag{21}$$

so this vector is also represented by the magenta arrow in Fig. 1. In this case, the outward normal vector actually does point outwards.

A similar argument shows that on the edge $x = 1$, both the outward pointing vector and the outward unit normal vector point in the same direction, drawn as the orange vector in Fig. 1.

Finally, suppose we add a path from $(t,x) = (0,0)$ to $(t,x) = (1,1)$, so we have a diagonal line. A vector $\vec{L}$ tangent to this line has equal components in the two directions, such as

$$\vec{L} = (1,1) \tag{22}$$

Thus the magnitude of this vector is

$$\left|\vec{L}\right|^2 = \eta_{\mu\nu} L^\mu L^\nu = -1 + 1 = 0 \tag{23}$$

Thus, any one-form of the form

$$\tilde{L} = (-1,1) \tag{24}$$

will be normal to $\vec{L}$, since

$$\tilde{L}\left(\vec{L}\right) = L_\nu L^\nu = -1 \times 1 + 1 \times 1 = 0 \tag{25}$$

and the corresponding normal vector is just $\vec{L}$ itself. In other words, for a light-like path, the tangents and normals are parallel to each other. Since the magnitudes of both the one-forms and normal vectors are always zero, we can't create a *unit* vector or one-form for a light-like path.