ONE-FORM BASIS

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The one-form is a tensor denoted by $\binom{0}{1}$. From the definition of a tensor, it is a function that takes one vector as an argument and returns a scalar. We've seen that the gradient of a scalar function can be represented by a one-form. To use the example from Schutz's book, suppose we have a scalar function $\phi(x)$ defined over a region of spacetime. We then consider some object moving through this region on some trajectory. The object's position can be written as a function of its proper time τ, so

$$\phi(\vec{x}) = \phi(\vec{x}(\tau)) \tag{1}$$

The position four-vector $\vec{x}$ can be written as a set of four functions of τ.

$$\vec{x} = [t(\tau), x(\tau), y(\tau), z(\tau)] \tag{2}$$

The rate of change of ϕ along the object's trajectory is the derivative of ϕ with respect to its proper time. Using the chain rule, we have

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau} \tag{3}$$

The second factor in each term is a component of the four-velocity $\vec{U}$:

$$\vec{U} = \left[\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right] \tag{4}$$

so we can write 3 as

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}U^t + \frac{\partial\phi}{\partial x}U^x + \frac{\partial\phi}{\partial y}U^y + \frac{\partial\phi}{\partial z}U^z \tag{5}$$

This expression satisfies the definition of a one-form, as it takes a single vector $\vec{U}$ as an argument and outputs a scalar $\frac{d\phi}{d\tau}$. From the action of a one-form on a vector which is given as

$$\tilde{p}\left(\vec{A}\right) = A^\alpha p_\alpha \tag{6}$$

where the p_α are the components of the one-form, we see that we can write 5 as

$$\frac{d\phi}{d\tau} = U^\alpha \tilde{d}\phi_\alpha \tag{7}$$

where $\tilde{d}\phi$ is the one-form with components

$$\tilde{d}\phi = \left[\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right] \tag{8}$$

We've also seen that a basis for one-forms can be written using the basis of four-vectors as

$$\tilde{\omega}^\alpha \left(\vec{e}_\beta\right) = \delta^\alpha{}_\beta \tag{9}$$

Note carefully that the α in $\tilde{\omega}^\alpha$ labels a complete one-form (and not just a single component), and the β in $\vec{e}_\beta$ labels a complete basis vector. This equation is saying that the β component of the one-form $\tilde{\omega}^\alpha$ is $\delta^\alpha{}_\beta$.

That is, the components of $\tilde{\omega}^\alpha$ in the coordinate system $\mathcal{O}$ must be

$$\begin{aligned} \tilde{\omega}^0 &\rightarrow (1,0,0,0) \\ \tilde{\omega}^1 &\rightarrow (0,1,0,0) \\ \tilde{\omega}^2 &\rightarrow (0,0,1,0) \\ \tilde{\omega}^3 &\rightarrow (0,0,0,1) \end{aligned} \tag{10}$$

Now consider the position four-vector in spacetime $\vec{x}$. Its four components are independent variables, so

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta \tag{11}$$

The LHS here is effectively the gradient of the position component x^α, so we can write it as

$$\tilde{d}x^\alpha = \left[\frac{\partial x^\alpha}{\partial x^0}, \frac{\partial x^\alpha}{\partial x^1}, \frac{\partial x^\alpha}{\partial x^2}, \frac{\partial x^\alpha}{\partial x^3}\right] \tag{12}$$

Only one of the components on the RHS is non-zero, because of 11.

11 is the same relation as 9. Thus we can take the basis one-forms to be the 'gradients' of the position four-vectors. That is

$$\tilde{\omega}^\alpha = \tilde{d}x^\alpha \tag{13}$$

Using this notation, we can write the gradient of an arbitrary scalar function f:

$$\tilde{d}f = \frac{\partial f}{\partial x^\alpha}\tilde{\omega}^\alpha = \frac{\partial f}{\partial x^\alpha}\tilde{d}x^\alpha \tag{14}$$

The notation can be a bit misleading if we interpret the $\tilde{d}f$ as indicating an infinitesimal or increment, which is not necessarily the case. The tilde over the $\tilde{d}$ indicates that the quantity is a one-form, and not necessarily an infinitesimal quantity. For example, referring back to 8, we have

$$\tilde{d}\phi = \frac{\partial \phi}{\partial x^0}\tilde{\omega}^0 + \frac{\partial \phi}{\partial x^1}\tilde{\omega}^1 + \frac{\partial \phi}{\partial x^2}\tilde{\omega}^2 + \frac{\partial \phi}{\partial x^3}\tilde{\omega}^3 \tag{15}$$

$$= \frac{\partial \phi}{\partial x^0}\tilde{d}x^0 + \frac{\partial \phi}{\partial x^1}\tilde{d}x^1 + \frac{\partial \phi}{\partial x^2}\tilde{d}x^2 + \frac{\partial \phi}{\partial x^3}\tilde{d}x^3 \tag{16}$$