ONE-FORMS IN POLAR COORDINATES

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Post date: 11 Dec 2022.

We'll consider coordinate transformations between rectangular and polar coordinates in two dimensions. This time we'll look at one-forms instead of vectors. Polar coordinates r and θ are defined in terms of rectangular coordinates by

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned} \tag{1}$$

and the inverse

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{2}$$

The radial coordinate r is the distance from the origin and the polar angle θ is measured counterclockwise from the horizontal axis.

We've examined how vectors transform from rectangular to polar coordinates, so here we'll do the same for one-forms.

A vector $\vec{V}$ can be converted to its corresponding one-form $\tilde{V}$ by using the metric tensor $g_{\alpha\beta}$ according to

$$\tilde{d}V_\alpha = g_{\alpha\beta}V^\alpha \tag{3}$$

The metric tensor in polar coordinates is

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{4}$$

The inverse is

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \tag{5}$$

Thus if we know the components of a vector V^α in one coordinate system, we can find the corresponding components V_α in the same coordinate system by using the metric tensor.

However, to convert from one system (rectangular, say) to another (polar), we need a transformation matrix as we had for transforming vectors. A typical one-form is the gradient of a function f, as in

$$\tilde{d}f=\left[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right] \tag{6}$$

To transform to polar coordinates, we use the chain rule:

$$\begin{aligned}\frac{\partial f}{\partial r}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\ \frac{\partial f}{\partial \theta}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}\end{aligned} \tag{7}$$

This can be written as a matrix equation:

$$\left[\begin{array}{cc}\frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta}\end{array}\right]=\left[\begin{array}{cc}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{array}\right]\left[\begin{array}{cc}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{array}\right] \tag{8}$$

Note that the one-forms are written as rows rather than columns. If we define

$$\Lambda^{\alpha}{}_{\beta'}=\left[\begin{array}{cc}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{array}\right] \tag{9}$$

where the primed indices refer to polar and the unprimed to rectangular coordinates, then the transformation is

$$\tilde{d}f_{\beta'}=\Lambda^{\alpha}{}_{\beta'}\tilde{d}f_{\alpha} \tag{10}$$

The matrix 9 is the inverse of the matrix we used in transforming vectors. For rectangular to polar conversion, we have

$$\Lambda^{\alpha}{}_{\beta'}=\left[\begin{array}{cc}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{array}\right] \tag{11}$$

Example 1. Consider the vector $\vec{W}=[1,1]$ in rectangular coordinates. The corresponding one-form in rectangular coordinates is the same, since the metric tensor in rectangular coordinates is the unit matrix.

$$\tilde{W}_{\text{rect}}=[1,1] \tag{12}$$

We saw earlier that the vector $\vec{W}$ in polar coordinates is:

$$\vec{W}_{\text{polar}}(r,\theta)=\left[\cos\theta+\sin\theta,\frac{1}{r}(-\sin\theta+\cos\theta)\right] \tag{13}$$

We can use 4 to find the one-form:

$$\tilde{W}_{\text{polar},\alpha} = g_{\alpha\beta} W^{\beta} \tag{14}$$
$$= [\cos\theta + \sin\theta, r(-\sin\theta + \cos\theta)] \tag{15}$$

We can also find the one-form by transforming directly from rectangular coordinates using 11:

$$\tilde{W}_{\text{polar},\beta'} = \Lambda^{\alpha}{}_{\beta'} \tilde{W}_{\text{rect},\alpha} \tag{16}$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{17}$$
$$= [\cos\theta + \sin\theta, r(-\sin\theta + \cos\theta)] \tag{18}$$

This gives the same result.

As mentioned when we analyzed vectors, the components of a one-form in polar coordinates depend on the point (r, θ) at which we use the basis one-forms. Schutz derives the basis one-forms in eqns 5.26 and 5.27:

$$\begin{aligned} \tilde{d}r &= \cos\theta\,\tilde{d}x + \sin\theta\,\tilde{d}y \\ \tilde{d}\theta &= -\frac{\sin\theta}{r}\tilde{d}x + \frac{\cos\theta}{r}\tilde{d}y \end{aligned} \tag{19}$$

We can write out 18 using these bases as

$$\tilde{W}_{\text{polar}} = (\cos\theta + \sin\theta)\,\tilde{d}r + r(-\sin\theta + \cos\theta)\,\tilde{d}\theta \tag{20}$$
$$\begin{aligned} &= (\cos\theta + \sin\theta)\left(\cos\theta\,\tilde{d}x + \sin\theta\,\tilde{d}y\right) + \\ &\quad r(-\sin\theta + \cos\theta)\left(-\frac{\sin\theta}{r}\tilde{d}x + \frac{\cos\theta}{r}\tilde{d}y\right) \end{aligned} \tag{21}$$

Multiplying this out and simplifying gives

$$\tilde{W}_{\text{polar}} = \tilde{d}x + \tilde{d}y \tag{22}$$

which agrees with 12.

Example 2. Consider the vector

$$\vec{V}_{\text{rect}} = \left[x^2 + 3y, y^2 + 3x\right] \tag{23}$$

We first write $\vec{V}$ in polar coordinates using 2:

$$\vec{V}_{\text{rect}} = \left[r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta\right] \tag{24}$$

As before, the one-form in rectangular coordinates has the same components.

$$\tilde{V}_{\text{rect}} = \left[r^2\cos^2\theta + 3r\sin\theta, r^2\sin^2\theta + 3r\cos\theta\right] \tag{25}$$

In polar coordinates this is

$$\begin{aligned}\vec{V}_{\text{polar}} = &\left[r^2\left(\cos^3\theta + \sin^3\theta\right) + 6r\sin\theta\cos\theta,\right.\\ &\left. r\left(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta\right) + 3\left(\cos^2\theta - \sin^2\theta\right)\right]\end{aligned} \tag{26}$$

We can multiply by the metric 4 to get the one-form (using the shorthand $c \equiv \cos\theta$ and $s \equiv \sin\theta$):

$$\tilde{V}_{\text{polar}} = \left[r^2\left(c^3+s^3\right) + 6rsc, r^3\left(cs^2 - c^2s\right) + 3r^2\left(c^2 - s^2\right)\right] \tag{27}$$

We can get the same result by starting with the rectangular form 24 and multiplying by 11:

$$\tilde{V}_{\text{polar}} = \tilde{V}_{\text{rect}}\Lambda^{\alpha}{}_{\beta'} \tag{28}$$

$$= \left[r^2\left(c^3+s^3\right) + 6rsc, r^3\left(cs^2 - c^2s\right) + 3r^2\left(c^2 - s^2\right)\right] \tag{29}$$

We can verify that if we write out $\tilde{V}_{\text{polar}}$ using 19 as

$$\tilde{V}_{\text{polar}} = \tilde{V}_{\text{polar},r}\tilde{d}r + \tilde{V}_{\text{polar},\theta}\tilde{d}\theta \tag{30}$$

and simplify (using Maple) we do indeed get $\tilde{V}_{\text{rect}}$ as given by 25.

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