PAINLEVÉ-GULLSTRAND (GLOBAL RAIN) COORDINATES

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It's time to take another look at the problems with the Schwarzschild metric at the event horizon.

The Schwarzschild metric (which we'll call the S metric) is:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

Problems arise when $r = 2GM$, which defines the event horizon. Let's take a step back and look at the two dimensional surface of a sphere, which can be described by the metric

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{2}$$

We can define the two angular coordinates θ and ϕ uniquely anywhere on the sphere except at the two poles. At that point $\theta = 0$ so $\sin\theta = 0$ and the $d\phi^2$ term disappears, no matter what ϕ is, so we can't assign a meaningful value to ϕ when $\theta = 0$, yet there is nothing special about the sphere at that point; the problem lies entirely with the coordinates we're using.

It looks as though the time coordinate t suffers the same fate in the S metric. When $r = 2GM$, the dt^2 term vanishes, so at this point we can't assign a meaningful value to t. As we showed in the earlier post, however, there doesn't appear to be anything special about the value $r = 2GM$, since the distance Δs and proper time interval $\Delta\tau$ are finite as we cross the event horizon. It therefore appears that the problem lies with the coordinates.

Another way of looking at it is to recall the relation between the S time coordinate t and the proper time τ:

$$d\tau = \sqrt{1-\frac{2GM}{r}}dt \tag{3}$$

The two times are equal at infinity, and for a given proper time interval, the corresponding interval dt gets larger and larger the closer we get to the event horizon, eventually becoming infinite at the horizon. Thus again we see that we can't assign a unique value to dt at the event horizon.

Moore (in his Chapter 14) also states that a particle that tries to be at rest at $r=2GM$ must have a light-like worldline (that is, $ds^2=0$), since if the particle is at rest, $dr=d\theta=d\phi=0$. However, if $r=2GM$, the metric component $g_{rr}=1/0$, so we're dividing zero by zero, which is indeterminate, so I wouldn't put too much weight on that argument.

In any event, as we saw earlier, when we cross the event horizon, the t and r coordinates swap their meanings, with the result that a particle must continuously move inward (in the direction of decreasing r) as time advances. In other words, it's impossible for a particle to remain at rest when it crosses the event horizon.

It's worth pointing out that we can still define a surface of constant r and t within the event horizon by setting $dr=dt=0$. This gives us the metric for the surface of a sphere in 2, so r still does define a radial distance in the sense that we can take the circumference of a shell within the event horizon and divide by 2π to get r. The key point is that a particle cannot be at rest on such a surface since the r coordinate is a timelike coordinate.

We therefore would like to find a coordinate system in which we *can* define a meaningful time at every point. However, merely changing coordinates cannot change any of the physical predictions that we got from the S metric. In particular, it must still be true that once we cross the event horizon, we are forced to keep travelling inwards. We would just like this to happen without having a singularity at $r=2GM$, and without the r and t coordinates swapping their meaning.

There are several alternative coordinate systems that achieve this, but we'll have a look at only one of them in this post: the Painlevé-Gullstrand system (sometimes called the *global rain* coordinate system; we'll refer to it as the PG system). The PG system introduces a new time coordinate $\mathring{t}$ (that's a t with a little circle on top to represent a raindrop), while keeping the r, θ and ϕ coordinates from the S metric. To avoid the problem of a time singularity at the event horizon, we use the following scheme. Along each radial line, we drop (from rest) a sequence of clocks in from the infinity towards the black hole. Since the clocks start from infinity, their proper time is equal to the S time when they are dropped. To find the PG time $\mathring{t}$ of some event at a given spatial location (that is, with given values of r, θ and ϕ) we find the clock that is falling down that particular radial line and passes the location r at the same time as the event occurs.

To work out the PG metric, we need to find $\mathring{t}$ in terms of the S coordinates so we can do the transformation. From symmetry, all directions in space are equivalent, so $\mathring{t}$ cannot depend on θ or ϕ, so it must be a function of t and r only: $\mathring{t}=\mathring{t}(t,r)$. We therefore seek the derivatives in the equation

$$d\mathring{t} = \frac{\partial \mathring{t}}{\partial t}dt + \frac{\partial \mathring{t}}{\partial r}dr \tag{4}$$

The idea is that if we can determine the two partial derivatives, we can solve this equation for dt in terms of $d\mathring{t}$ and dr and substitute the result into the S metric to get the PG metric.

Let's consider the derivative $\partial \mathring{t}/\partial t$ first. This term occurs when $dr = 0$, that is, when we look at two events that occur at the same spatial location but are separated by the S time interval dt. In terms of the PG system, $\mathring{t}$ is measured as the proper time on a clock falling in on a given radial line. The only way we can get two such clocks passing the same point but separated by a time dt is if they were dropped at slightly different times from the same point. Since the paths followed by the two clocks are identical in space (and we're assuming that the system is static, so the black hole isn't moving), they must take the same time to traverse these paths. In the S system, the time at which clock i ($i = 1, 2$) arrives at the given point is

$$t_i = t_{i;0} + \Delta t \tag{5}$$

where $t_{i;0}$ is the time at which clock i was dropped, and Δt is the time taken (as measured in the S system) to travel to the given point from the point at which it was dropped, and this time is the same for both clocks. Therefore

$$dt = t_2 - t_1 = t_{2;0} - t_{1;0} \tag{6}$$

That is, the time interval between the events of the clocks being dropped and between the events of the clocks' arrival at the point where the two events occur (as measured in the S system) are the same.

Now consider the PG system. Here, the time interval between the dropping of the clocks is the same as in the S system, since the clocks are synchronized with the S system when they are dropped. In the PG system, however, the time $\mathring{t}$ is the proper time registered by the clocks as they fall. However, since the two paths are identical, both the *proper* times taken by the two clocks must be same as well. Thus we have

$$\mathring{t}_i = t_{i;0} + \Delta\tau = t_{i;0} + \Delta\mathring{t} \tag{7}$$

so the increment in PG time between the two events is

$$d\mathring{t} = \mathring{t}_2 - \mathring{t}_1 = t_{2;0} - t_{1;0} = dt \tag{8}$$

That is:

$$\frac{\partial \mathring{t}}{\partial t} = 1 \tag{9}$$

At first glance, we might think that this implies $\mathring{t} = t$, and that the PG system is identical to the S system. However, that would be true only if $\partial\mathring{t}/\partial r = 0$ which, as we'll now see, isn't true. To show this, we return to the equations of motion for a particle in S coordinates. For a clock dropped from rest at infinity, the energy per unit mass is $e = 1$ and the angular momentum is $\ell = 0$, so we get

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \tag{10}$$

where the minus sign indicates that we're moving inwards (decreasing r). For the S time, we get

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{11}$$

so

$$\frac{dr}{dt} = -\sqrt{\frac{2GM}{r}}\left(1 - \frac{2GM}{r}\right) \tag{12}$$

To work out $\partial\mathring{t}/\partial r$ we need to consider two events that occur at the same S time ($dt = 0$) but at slightly different radii. The clock that arrives at the smaller radius must have been dropped slightly earlier so that it has the extra time to fall the extra distance. However, the PG time (= proper time) to fall the extra distance is not equal to the S time to fall that same distance, so there will be *two* contributions to $\partial\mathring{t}/\partial r$: one because one clock was dropped earlier than the other, and a second due to the fact that one clock falls further than the other.

First, how much extra S time is required to fall the extra distance dr? From 12, this depends on the radius, so

$$dt = -\left[\sqrt{\frac{2GM}{r}}\left(1 - \frac{2GM}{r}\right)\right]^{-1} dr \tag{13}$$

That is, the clock that ends up at the lower radius must be dropped a time dt (in S time) before the other clock. Since both time systems are synchronized when the clocks are dropped, this also represents the difference in PG time due to the different dropping times of the two clocks. In order to get the sign right, we're defining dt to be as in 6 above, that is, the time at which the second clock is dropped minus that at which the first clock is dropped.

Now suppose we dropped the two clocks at the *same* time, but let one fall a distance dr further than the other. We need to calculate the difference in *proper* (that is, PG) time for these two trajectories. According to 10, this is

$$d\tau = \sqrt{\frac{r}{2GM}} dr \tag{14}$$

Note that we've dropped the minus sign, since we're finding the difference in proper time required for the *second* clock minus that for the first clock. The second clock falls a shorter distance, so it takes less time, so $d\tau < 0$ here (remember that $dr < 0$).

The total difference in PG time $d\mathring{t}$ is the sum of these two effects. That is:

$$d\mathring{t} = \sqrt{\frac{r}{2GM}} \left[1 - \left(1 - \frac{2GM}{r} \right)^{-1} \right] dr \tag{15}$$

$$= -\sqrt{\frac{r}{2GM}} \frac{2GM}{r - 2GM} dr \tag{16}$$

$$= -\sqrt{\frac{2GM}{r}} \frac{1}{1 - 2GM/r} dr \tag{17}$$

$$= \sqrt{\frac{2GM}{r}} \frac{1}{1 - 2GM/r} |dr| \tag{18}$$

(As an aside at this point, note that if we apply the same logic to the S system, then the time difference at the dropping of the two clocks is given by 13, while the difference in the times required to fall the two different distances is just the negative of 13, giving a net $dt = 0$, which is what we're assuming here.)

Thus, since $\partial\mathring{t}/\partial r$ is the rate of change of $\mathring{t}$ with respect to *increasing* r, we have

$$\frac{\partial \mathring{t}}{\partial r} = \sqrt{\frac{2GM}{r}} \frac{1}{1 - 2GM/r} \tag{19}$$

and the differential is

$$d\mathring{t} = dt + \sqrt{\frac{2GM}{r}} \frac{1}{1 - 2GM/r} dr \tag{20}$$

$$dt = d\mathring{t} - \sqrt{\frac{2GM}{r}} \frac{1}{1 - 2GM/r} dr \tag{21}$$

We can substitute this back into the S metric 1 and get

$$\begin{aligned}
ds^2 &= -\left(1-\frac{2GM}{r}\right)\left(d\mathring{t}-\sqrt{\frac{2GM}{r}}\frac{1}{1-2GM/r}dr\right)^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+ \\
&\quad r^2d\theta^2+r^2\sin^2\theta d\phi^2 && (22)\\
&= -\left(1-\frac{2GM}{r}\right)d\mathring{t}^2+2\sqrt{\frac{2GM}{r}}d\mathring{t}\,dr+\left(1-\frac{2GM}{r}\right)\left(1-\frac{2GM}{r}\right)^{-1}dr^2+ \\
&\quad r^2d\theta^2+r^2\sin^2\theta d\phi^2 && (23)\\
&= -\left(1-\frac{2GM}{r}\right)d\mathring{t}^2+2\sqrt{\frac{2GM}{r}}d\mathring{t}\,dr+dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 && (24)
\end{aligned}$$

The last equation gives the PG metric, which we can see is *not* diagonal, due to the $d\mathring{t}\,dr$ term.

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