

## PAINLEVÉ-GULLSTRAND METRIC - PHOTON PATHS INSIDE THE EVENT HORIZON

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The Painlevé-Gullstrand (PG, also known as the global rain system) metric is a different way of describing empty space surrounding a spherically symmetric mass that avoids some of the problems encountered with the Schwarzschild metric (S metric). The PG metric is

$$ds^2 = - \left( 1 - \frac{2GM}{r} \right) d\hat{t}^2 + 2\sqrt{\frac{2GM}{r}} d\hat{t} dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

where  $\hat{t}$  is the PG time coordinate, and is the proper time as measured by a clock falling along a radial line from rest at infinity. The  $\hat{t}$  coordinate, since it is measured as the proper time on a clock, is always a time coordinate, so the PG metric avoids the switch between distance and time coordinates that occurs in the S metric. It might seem that this causes problems for  $r < 2GM$ , since there the metric component  $g_{\hat{t}\hat{t}} = - \left( 1 - \frac{2GM}{r} \right)$  becomes positive, so it might seem that all the terms are positive, resulting in  $ds^2 > 0$  for a time-like world line (contrary to what it should be). However, the off-diagonal metric component  $g_{\hat{t}r} = 2\sqrt{\frac{2GM}{r}}$  comes to the rescue here, since for  $r < 2GM$ , if we have  $dr < 0$  we can still get  $ds^2 < 0$ . That is, for a time-like world line, we must have  $dr < 0$  once we cross the event horizon, which is the same condition that we got from the S metric, except there it was done by swapping the meanings of  $r$  and  $t$ .

For a radial world line  $d\theta = d\phi = 0$  and we have

$$ds^2 = - \left( 1 - \frac{2GM}{r} \right) d\hat{t}^2 + 2\sqrt{\frac{2GM}{r}} d\hat{t} dr + dr^2 \quad (2)$$

which we can write as

$$\left( \frac{ds}{d\hat{t}} \right)^2 = - \left( 1 - \frac{2GM}{r} \right) + 2\sqrt{\frac{2GM}{r}} \frac{dr}{d\hat{t}} + \left( \frac{dr}{d\hat{t}} \right)^2 \quad (3)$$

Photons travelling radially will have  $\left(\frac{ds}{dt}\right)^2 = 0$ , so we can find  $\frac{dr}{dt}$  by solving the quadratic equation on the RHS.

$$\frac{dr}{dt} = \frac{1}{2} \left[ -2\sqrt{\frac{2GM}{r}} \pm \sqrt{\frac{8GM}{r} + 4\left(1 - \frac{2GM}{r}\right)} \right] \quad (4)$$

$$= -\sqrt{\frac{2GM}{r}} \pm 1 \quad (5)$$

The minus sign corresponds to an inward-moving photon, and the plus sign to an outward-moving one. Leaving aside the problem that, for  $r < 2GM$ , both these speeds can exceed 1 in magnitude, we can solve this equation to find  $\dot{t}$  as a function of  $r$ . We get, for the inward-moving photon

$$-\int \frac{dr}{\sqrt{\frac{2GM}{r}} + 1} = \int dt \quad (6)$$

Using Maple for the integral, we get

$$\dot{t} = -r + 2\sqrt{2GM}r - 4GM \ln \left[ \sqrt{2GM} + \sqrt{r} \right] + C \quad (7)$$

where  $C$  is the constant of integration. If we start the photon at  $r = 2GM$  at  $\dot{t} = 0$ , then

$$C = -2GM + 4GM \ln \left( 2\sqrt{2GM} \right) \quad (8)$$

$$\frac{\dot{t}}{GM} = 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4 \ln \left[ \sqrt{2GM} + \sqrt{r} \right] + 4 \ln \left( 2\sqrt{2GM} \right) \quad (9)$$

$$= 2\sqrt{\frac{2r}{GM}} - \frac{r}{GM} - 2 - 4 \ln \left[ \frac{1}{2} \left( 1 + \sqrt{\frac{r}{2GM}} \right) \right] \quad (10)$$

When  $r = 0$ , we get

$$\frac{\dot{t}}{GM} = -2 + 4 \ln 2 = 0.7726 \quad (11)$$

A plot of  $\frac{\dot{t}}{GM}$  versus  $\frac{r}{GM}$  is as shown in Fig. 1.

The particle starts off at  $r/GM = 2$  and its world line travels up and to the left until it hits  $r = 0$ .

For an outward moving photon, we get

$$-\int \frac{dr}{\sqrt{\frac{2GM}{r}} - 1} = \int dt \quad (12)$$

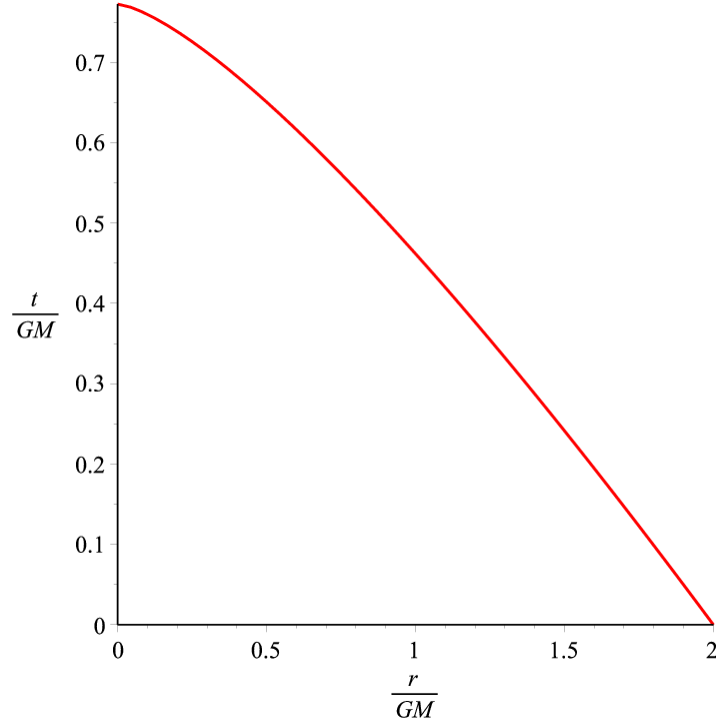


FIGURE 1. Inward-moving photon.

This time, we get for the integral

$$\dot{t} = r + 2\sqrt{2GM}r + 4GM \ln \left[ \sqrt{2GM} - \sqrt{r} \right] + C \quad (13)$$

In this case, we can't start an outward-moving photon at  $r = 2GM$ , since this gives an infinite logarithm. In fact, if we emit an outward-moving photon at some  $r < 2GM$ , it will actually move *inward* (since we require  $dr < 0$ ), although if we emit the photon near the event horizon (from the inside), it will take a very long time to move inwards from its starting point. To make things definite, suppose  $\dot{t} = 0$  at  $r = GM$ . Then

$$C = - \left( 1 + 2\sqrt{2} \right) GM - 4GM \ln \left( \left( \sqrt{2} - 1 \right) \sqrt{GM} \right) \quad (14)$$

$$\frac{\dot{t}}{GM} = \frac{r}{GM} + 2\sqrt{\frac{2r}{GM}} - \left( 1 + 2\sqrt{2} \right) + 4 \ln \left[ \frac{\sqrt{2} - \sqrt{r/GM}}{\sqrt{2} - 1} \right] \quad (15)$$

A plot is in Fig. 2

Again, the motion along the red curve is up and to the left. A photon emitted outward at  $r = GM$  takes  $\dot{t} = 1.0833GM$  to reach  $r = 0$ , while an

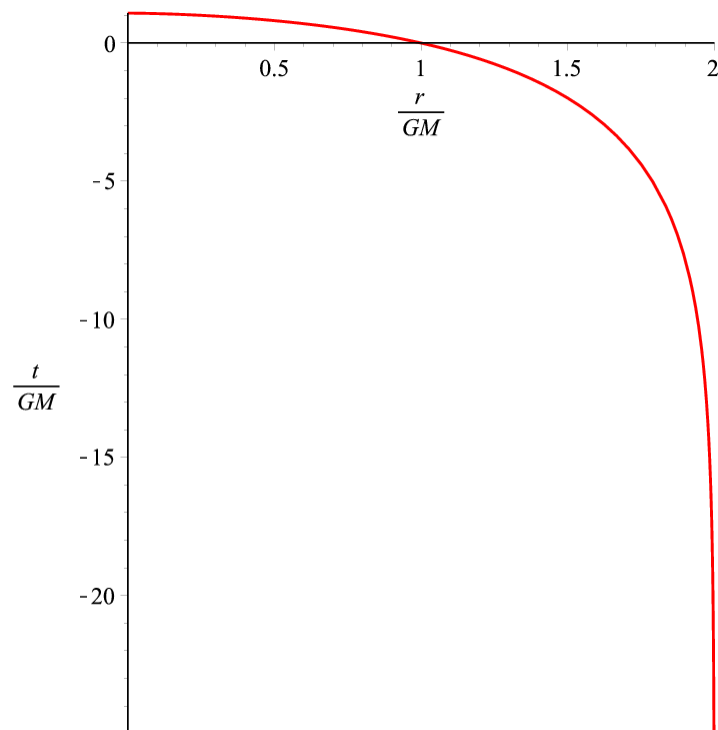


FIGURE 2. Outward-moving photon.

inward moving photon takes  $\dot{t} = 0.3108GM$  (which we get by evaluating  $\dot{t}$  for  $r = GM$  in 7 and taking the difference between that and 0.7726) .