PARALLEL TRANSPORT AND THE GEODESIC EQUATION

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Parallel transport of a vector along a curve occurs when the vector $\vec{V}$ is moved along the curve in such a way that its orientation and length are the same at neighbouring infinitesimal points along the curve. If the curve is parameterized by the variable λ, then the condition for parallel transport of a vector $\vec{V}$ is

$$\frac{d\vec{V}}{d\lambda}=0 \tag{1}$$

In components, this is

$$\frac{\partial V^{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\lambda}=\frac{\partial V^{\mu}}{\partial x^{\nu}}U^{\nu}=U^{\nu}V^{\mu}{}_{,\nu}=0 \tag{2}$$

where the vector $\vec{U}(\lambda)$ with components

$$U^{\nu}(\lambda)\equiv\frac{dx^{\nu}}{d\lambda} \tag{3}$$

is the tangent vector to the curve.

In relativity, we assume that at any given point $\mathcal{P}$ the spacetime is locally flat. In flat spacetime, it is always possible to find a coordinate system in which the Christoffel symbols are all zero. In such a coordinate system, The ordinary derivative in 2 is the same as the covariant derivative, so we have

$$U^{\nu}V^{\mu}{}_{;\nu}=0 \tag{4}$$

Since both U^{ν} and $V^{\mu}{}_{;\nu}$ are tensors, and if the components of a tensor are all zero in one coordinate system, then they are also zero in all coordinate systems, then 4 is true in systems where the Christoffel symbols are not zero, so we have

$$U^{\nu}V^{\mu}{}_{;\nu}=U^{\nu}\left(V^{\mu}{}_{,\nu}+\Gamma^{\mu}{}_{\nu\alpha}V^{\alpha}\right)=0 \tag{5}$$

This is the condition for parallel transport of an arbitrary vector $\vec{V}$ along a curve. A geodesic may be defined as a curve where its own tangent vector is parallelly transported along the curve. That is, it is a curve where $\vec{V}=\vec{U}$, where $\vec{U}$ is the tangent vector to the curve. In this case, 5 becomes

$$U^{\nu}U^{\mu}{}_{,\nu}+U^{\nu}U^{\alpha}\Gamma^{\mu}{}_{\nu\alpha}=0 \tag{6}$$

We now use

$$U^{\nu}=\frac{dx^{\nu}}{d\lambda} \tag{7}$$

$$U^{\mu}{}_{,\nu}=\frac{\partial}{\partial x^{\nu}}\left(\frac{dx^{\mu}}{d\lambda}\right) \tag{8}$$

so we have

$$U^{\nu}U^{\mu}{}_{,\nu}+U^{\nu}U^{\alpha}\Gamma^{\mu}{}_{\nu\alpha}=\frac{dx^{\nu}}{d\lambda}\frac{\partial}{\partial x^{\nu}}\left(\frac{dx^{\mu}}{d\lambda}\right)+\Gamma^{\mu}{}_{\nu\alpha}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\alpha}}{d\lambda} \tag{9}$$

We can condense the first term by using

$$\frac{dx^{\nu}}{d\lambda}\frac{\partial}{\partial x^{\nu}}=\frac{d}{d\lambda} \tag{10}$$

so we have

$$\boxed{\frac{d^{2}x^{\mu}}{d\lambda^{2}}+\Gamma^{\mu}{}_{\nu\alpha}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\alpha}}{d\lambda}=0} \tag{11}$$

This is the *geodesic equation*, which is satisfied by any geodesic curve. In 11, the $\Gamma^{\mu}{}_{\nu\alpha}$ are all known, as they can be calculated from the metric, so this equation is actually a set of 4 ordinary (as opposed to partial) differential equations for the components x^{μ} of the geodesic curve.

We previously derived the geodesic equation by maximizing the proper time between two events, using the Lagrangian. We found the geodesic equation there in the form

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^{\mu}}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{\nu\mu}}{\partial x^{\alpha}}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau}=0 \tag{12}$$

At first glance, it might seem that is a completely different equation from 11, but they are in fact the same, as we can show.

First, we take the parameter λ in 11 to be the proper time τ, since the proper time is the appropriate parameter to use for a particle's world line, and we want to show that a particle follows a geodesic path.

Next, consider the first term in 12 which can write as

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^{\mu}}{d\tau}\right) = \frac{dg_{\alpha\mu}}{d\tau}\frac{dx^{\mu}}{d\tau} + g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2} \tag{13}$$

$$= \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau} + g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2} \tag{14}$$

Substituting this back into 12 we have, using the comma notation for derivatives of the metric:

$$g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2} + g_{\alpha\mu,\nu}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau} - \frac{1}{2}g_{\nu\mu,\alpha}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau} = 0 \tag{15}$$

In the second term, μ and ν are dummy variables, so we can swap them to get

$$g_{\alpha\mu,\nu}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau} = \frac{1}{2}\left(g_{\alpha\mu,\nu}\frac{dx^{\nu}}{d\tau}\frac{dx^{\mu}}{d\tau} + g_{\alpha\nu,\mu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}\right) \tag{16}$$

We therefore have

$$g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2} + \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\nu\mu,\alpha}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{17}$$

Finally, we can multiply through by $g^{\alpha\beta}$ and use $g^{\alpha\beta}g_{\alpha\mu} = \delta^{\beta}{}_{\mu}$ to get

$$\frac{d^2x^{\beta}}{d\tau^2} + \frac{1}{2}g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\nu\mu,\alpha}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{18}$$

We can now relabel the indices as follows: $\beta \to \mu$, $\alpha \to \sigma$, $\mu \to \alpha$ to get

$$\frac{d^2x^{\mu}}{d\tau^2} + \frac{1}{2}g^{\sigma\mu}\left(g_{\sigma\alpha,\nu} + g_{\sigma\nu,\alpha} - g_{\nu\alpha,\sigma}\right)\frac{dx^{\alpha}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0 \tag{19}$$

We now recall the formula for the Christoffel symbols in terms of the metric, and we find that

$$\Gamma^{\mu}{}_{\nu\alpha} = \frac{1}{2}g^{\sigma\mu}\left(g_{\sigma\alpha,\nu} + g_{\sigma\nu,\alpha} - g_{\nu\alpha,\sigma}\right) \tag{20}$$

This shows that 11 and 12 are actually the same equation.

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