PARALLEL TRANSPORT AND THE SCALAR PRODUCT

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 6 Jan 2023.

Parallel transport of a vector along a curve occurs when the vector $\vec{V}$ is moved along the curve in such a way that its orientation and length are the same at neighbouring infinitesimal points along the curve. If the curve is parameterized by the variable λ, then the condition for parallel transport of a vector $\vec{V}$ is

$$\frac{d\vec{V}}{d\lambda}=0 \tag{1}$$

In components, this is

$$\frac{\partial V^{\mu}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\lambda}=\frac{\partial V^{\mu}}{\partial x^{\nu}}U^{\nu}=0 \tag{2}$$

where the vector $\vec{U}(\lambda)$ with components

$$U^{\nu}(\lambda)\equiv\frac{dx^{\nu}}{d\lambda} \tag{3}$$

is the tangent vector to the curve.

Suppose we have two vectors $\vec{A}$ and $\vec{B}$ and we parallel transport them along a curve. It turns out that the scalar product

$$\vec{A}\cdot\vec{B}=g_{\mu\nu}A^{\mu}B^{\mu} \tag{4}$$

is constant along the curve. To see this, we take the derivative of 4 with respect to λ:

$$\frac{d}{d\lambda}\left(g_{\mu\nu}A^{\mu}B^{\mu}\right)=\left[g_{\mu\nu,\alpha}A^{\mu}B^{\nu}+g_{\mu\nu}A^{\mu}{}_{,\alpha}B^{\nu}+g_{\mu\nu}A^{\mu}B^{\nu}{}_{,\alpha}\right]\frac{dx^{\alpha}}{d\lambda} \tag{5}$$

$$=g_{\mu\nu,\alpha}A^{\mu}B^{\nu}U^{\alpha}+g_{\mu\nu}A^{\mu}{}_{,\alpha}U^{\alpha}B^{\nu}+g_{\mu\nu}A^{\mu}B^{\nu}{}_{,\alpha}U^{\alpha} \tag{6}$$

The second and third terms are zero because of 2. The first term is zero because of the local flatness theorem, which states that at any point we can write the metric tensor $g_{\mu\nu}$ that is equal to the flat spacetime metric $\eta_{\mu\nu}$ up to first order, so that $g_{\mu\nu,\alpha}=0$ for all μ,ν,α (although second and higher derivatives may not be zero).

Thus we have shown that, along the curve with parameter λ,

$$\frac{d}{d\lambda}\left(g_{\mu\nu}A^{\mu}B^{\mu}\right)=\frac{d}{d\lambda}\left(\vec{A}\cdot\vec{B}\right)=0 \tag{7}$$

Qualitatively, this is reasonable, since the idea behind parallel transport is that a vector is transported along the curve so that its length remains constant, and its orientation relative to the curve is also constant. Thus if we transport two vectors this way, we'd expect their scalar product to be a constant.

One consequence of this is that the nature of a geodesic curve retains its spacelike, timelike or null property over its entire length. One way of defining a geodesic as a curve that is obtained by parallel transport of its own tangent vector. In flat space, for example, a straight line's tangent is a vector parallel (in the Euclidean sense) to the straight line, and if we parallel transport this vector, we just continue the straight line in the same direction.

In curved space, we use this as a definition of a geodesic. Thus the vector $\vec{U}$ in the above discussion is a tangent to a geodesic. The property of the geodesic at some point $\mathcal{P}$ is then defined by

$$\vec{U}\cdot\vec{U}\begin{cases}<0 & \text{timelike}\\ =0 & \text{null or lightlike}\\ >0 & \text{spacelike}\end{cases} \tag{8}$$

As the scalar product does not vary for a vector parallel transported along a curve, the value of $\vec{U}\cdot\vec{U}$ remains constant along a geodesic, so its nature is constant over its entire length.