PARALLEL TRANSPORT AROUND A SPHERICAL TRIANGLE

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Parallel transport of a vector along a curve occurs when the vector $\vec{V}$ is moved along the curve in such a way that its orientation and length are the same at neighbouring infinitesimal points along the curve. In flat space, this amounts to merely moving the vector parallel to itself along any curve, so that if the vector is eventually returned to its starting point, it will have the same direction as it started with.

In curved space, this is no longer true. The standard example is to consider the parallel transport of a vector around a spherical triangle. For example, if we start with a vector pointing east at latitude 0 and longitude 0 on the Earth and transport the vector northwards along the longitude 0 line, the vector remains pointing east at all points along this curve until it reaches the north pole. If we then transport the vector southwards along the longitude 90° east line, the vector now points due south along this curve. If, upon reaching the equator again, we transport the vector west along the equator, it will still be pointing south. When we return to the starting point, the vector has now rotated from east (when it started) to south, so it has rotated through 90°.

A theorem from spherical trigonometry states that the sum of the angles in any spherical triangle is greater than π, or 180°. We can find a relation between this sum of angles and the amount by which a vector is rotated when transported around a spherical triangle.

We consider a spherical triangle with vertices A, B and C, and start with a vector $\vec{V}$ at vertex A that makes an angle θ with the side AB. As we move the vector along the side towards vertex B, it always makes the same angle θ with this side as in Fig. 1. In the figure, side AB is part of the great circle shown in green, and side BC is part of the great circle shown in blue. The angle at vertex B is α.

When the vector reaches B, we now wish to transport it down the blue side towards vertex C. From the figure, we see that the vector makes an angle $\theta-\alpha$ with side BC, which it maintains as we move along side BC.

When the vector reaches vertex C (Fig. 2), we now want to move it along side CA back to vertex A. If the angle at vertex C is β, then from the figure we see that the (counterclockwise) angle the vector makes with side CA is $\pi-\beta+\theta-\alpha$.

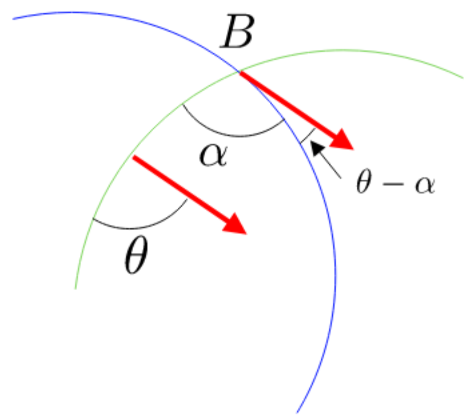

FIGURE 1. Vector (in red) moves along side AB until it reaches vertex B.

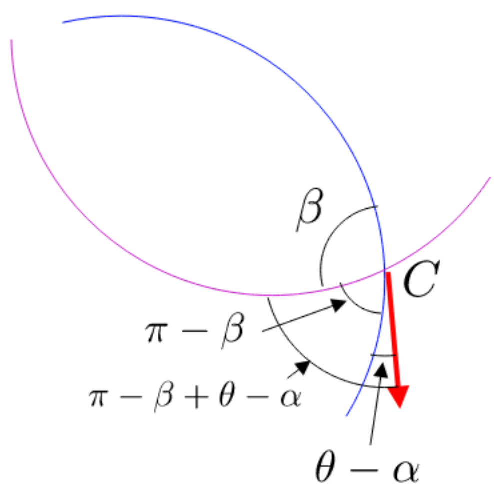

FIGURE 2. Vector reaches vertex C.

We now arrive back at vertex A (Fig. 3). It still makes an angle of $\pi - \beta + \theta - \alpha$ with the side CA. From the figure, we see that the counterclockwise angle the vector makes with side AB is now

$$(\pi - \gamma) + \gamma + (\pi - \gamma) + (\pi - \beta + \theta - \alpha) = 3\pi + \theta - \alpha - \beta - \gamma \quad (1)$$

This angle is equivalent to $\pi + \theta - \alpha - \beta - \gamma$ since we can subtract 2π (a complete circle) from the angle in 1. Thus the change in orientation of the vector due to the transport around the spherical triangle is

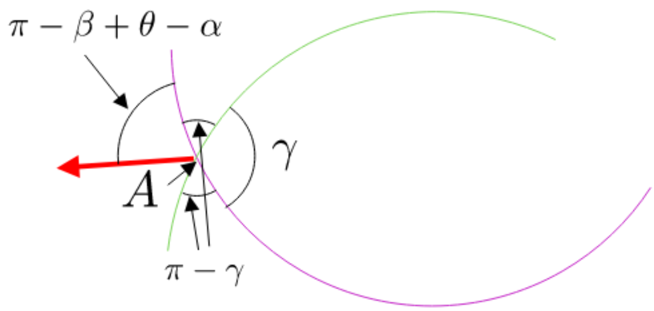

FIGURE 3. Vector arrives back at vertex A.

$$\Delta\theta = \theta-(\pi+\theta-\alpha-\beta-\gamma) \tag{2}$$
$$= \alpha+\beta+\gamma-\pi \tag{3}$$

That is, the amount that the vector is rotated is equal to the excess over π or 180° of the sum of the angles in a spherical triangle.

As a check, we note that for the example of the parallel transport over the Earth that we gave above, all angles were equal to $\frac{\pi}{2}$ or 90°, so $\Delta\theta=\frac{\pi}{2}$.

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