PARTICLE FALLING TOWARDS A MASS - TWO TYPES OF VELOCITY

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 2 Sep 2022.
The radial equation of motion in the Schwarzschild metric is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right)=\frac{1}{2}\left(e^2-1\right) \tag{1}$$

We can use this to derive an equation for dr/dt, the rate of change of r with respect to the Schwarzschild time coordinate. The coordinate t isn't the time as measured by any particular object (that time is the proper time τ in the reference frame of the object) so we wouldn't expect it to be the same as $dr/d\tau$.

To get the equation, we can use

$$\frac{dr}{d\tau}=\frac{dr}{dt}\frac{dt}{d\tau} \tag{2}$$

$$=\frac{dr}{dt}e\left(1-\frac{2GM}{r}\right)^{-1} \tag{3}$$

where the last line uses the definition of e. Plugging this into the top equation we get

$$\frac{dr}{dt}=\frac{1}{e}\left(1-\frac{2GM}{r}\right)\left[e^2-1+2GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right)-\frac{\ell^2}{r^2}\right]^{1/2} \tag{4}$$

As r approaches $2GM$, $dr/dt\to 0$.

In the special case where we drop an object from rest at $r=r_0$, we can work out both dr/dt and $dr/d\tau$. In this case, motion is radially inward so $\ell=0$. To find e, we use the fact that for an object at rest at $r=r_0$:

$$e = \left(1-\frac{2GM}{r_0}\right)\frac{dt}{d\tau} \tag{5}$$
$$= \left(1-\frac{2GM}{r_0}\right)u^t \tag{6}$$
$$= \left(1-\frac{2GM}{r_0}\right)\left(1-\frac{2GM}{r_0}\right)^{-1/2} \tag{7}$$
$$= \left(1-\frac{2GM}{r_0}\right)^{1/2} \tag{8}$$

We have therefore

$$\frac{dr}{dt} = \frac{1}{e}\left(1-\frac{2GM}{r}\right)\left(e^2-1+\frac{2GM}{r}\right)^{1/2} \tag{9}$$
$$= \frac{1-\frac{2GM}{r}}{\sqrt{1-\frac{2GM}{r_0}}}\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)} \tag{10}$$
$$= \left(1-\frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}}\sqrt{\frac{r_0-r}{r_0-2GM}} \tag{11}$$

From 1 we get

$$\frac{dr}{d\tau} = \sqrt{e^2-1+\frac{2GM}{r}} \tag{12}$$
$$= \sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_0}\right)} \tag{13}$$

Comparing the two, we see that

$$\frac{dr}{dt} = \frac{1-\frac{2GM}{r}}{\sqrt{1-\frac{2GM}{r_0}}}\frac{dr}{d\tau} \tag{14}$$

For the case where the object is released from rest at $r_0 = \infty$, the speed at $r = 6GM$ is

$$\frac{dr}{d\tau} = \frac{1}{\sqrt{3}} \tag{15}$$

which agrees with the earlier calculation done by a different method.