PARTICLE ORBITS - CONSERVED QUANTITIES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 30 Aug 2022.

One form of the geodesic equation is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{1}$$

A geodesic is the path followed by a free particle in a given metric, so we can apply this equation to the Schwarzschild metric to discover what orbits a particle can have around a mass. The metric is

$$ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{2}$$

From the t component, we get from 1:

$$\frac{d}{d\tau}\left(g_{tj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial t}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{3}$$

Since the metric does not depend on t the second term is zero. Also, since the metric is diagonal, the first term becomes

$$\frac{d}{d\tau}\left(g_{tj}\frac{dx^j}{d\tau}\right)=\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) \tag{4}$$

$$=-\frac{d}{d\tau}\left(\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}\right) \tag{5}$$

$$=0 \tag{6}$$

Note that r will in general depend on the proper time τ as the particle orbits the mass, so we can't take $1-\frac{2GM}{r}$ outside the derivative in the second line.

Integrating with respect to τ, we get

$$\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}=e \tag{7}$$

where e is a constant of integration.

The meaning of e can be inferred from the following argument. At $r=\infty$, the relation becomes

$$\frac{dt}{d\tau} = e \tag{8}$$

We've seen that the Schwarzschild t coordinate is the same as the object's proper time for an object at rest, when measured by an observer at infinity. Thus for an object at rest, $e = 1$. However, in general $dt/d\tau = u^t$, the time component of the four-velocity. In this case, t is the time as measured by an observer at rest at infinity, and τ is the proper time as measured by the object, which may be moving. The four-momentum's time component is the energy, and the four-velocity is the four-momentum per unit mass, so e is the energy per unit mass of the object which remains constant as the object moves in from infinity.

Now look at the ϕ component of 1. Again, since the metric does not depend on ϕ and it is diagonal, we get

$$\frac{d}{d\tau}\left(g_{\phi j}\frac{dx^j}{d\tau}\right) = \frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) \tag{9}$$

$$= -\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) \tag{10}$$

$$= 0 \tag{11}$$

$$r^2\sin^2\theta\frac{d\phi}{d\tau} = \ell \tag{12}$$

where ℓ is another constant. If we look at motion in the equatorial plane, $\theta = \pi/2$ and $d\phi/d\tau \equiv \omega$ which is the angular speed of rotation, so we get

$$r^2\omega = \ell \tag{13}$$

which is equivalent to the classical definition if ℓ is the angular momentum.

In fact, any initial velocity lies in *some* equatorial plane (we can always redefine the location of the z axis so that the velocity lies in the plane with $\theta = \pi/2$) so for any specific motion, we can assume $\theta = \pi/2$ without, as they say, any loss of generality. Also, because the metric is spherically symmetric, any motion that starts in an equatorial plane must stay in the plane, since there is no asymmetry that would push the object to one side or the other of that plane.

Looking at the θ component of 1, we can see that an equatorial path is a geodesic:

$$\frac{d}{d\tau}\left(g_{\theta j}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\partial_\theta g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=\frac{d}{d\tau}\left(g_{\theta\theta}\frac{d\theta}{d\tau}\right)-\frac{1}{2}\left(2r^2\sin\theta\cos\theta\right)\left(\frac{d\phi}{d\tau}\right)^2 \tag{14}$$

$$=2r\frac{dr}{d\tau}\frac{d\theta}{d\tau}+r^2\frac{d^2\theta}{d\tau^2}-\frac{1}{2}\left(2r^2\sin\theta\cos\theta\right)\left(\frac{d\phi}{d\tau}\right)^2 \tag{15}$$

$$=0 \tag{16}$$

If $\theta=\pi/2=$ constant, then all three terms in the second line are zero, so the geodesic equation is satisfied. Note that this last conclusion doesn't *prove* that all geodesics lie in an equatorial plane; it merely verifies that a planar path *is* a solution. We rely on the symmetry argument to state that all geodesics must lie in an equatorial plane. In a different metric where the mass is not spherically symmetric, non-planar paths are possible.

Finally, we consider the r component of 1. Since all four components of the metric depend on r, this takes a bit more work. We can start with the universal relation for the four-velocity $\mathbf{u}\cdot\mathbf{u}=-1$, which in this metric is

$$g_{tt}\left(\frac{dt}{d\tau}\right)^2+g_{rr}\left(\frac{dr}{d\tau}\right)^2+g_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2+g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2=-1 \tag{17}$$

From above, we know that, since $\theta=\pi/2$, using 7 and 12:

$$\frac{dt}{d\tau}=e\left(1-\frac{2GM}{r}\right)^{-1} \tag{18}$$

$$\frac{d\theta}{d\tau}=0 \tag{19}$$

$$\frac{d\phi}{d\tau}=\frac{\ell}{r^2\sin^2\theta}=\frac{\ell}{r^2} \tag{20}$$

Therefore

$$-1=-\left(1-\frac{2GM}{r}\right)\left[e\left(1-\frac{2GM}{r}\right)^{-1}\right]^{2}+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^{2}+r^{2}\left(\frac{\ell}{r^{2}}\right)^{2} \tag{21}$$

$$-\left(1-\frac{2GM}{r}\right)=-e^{2}+\left(\frac{dr}{d\tau}\right)^{2}+\left(1-\frac{2GM}{r}\right)\frac{\ell^{2}}{r^{2}} \tag{22}$$

$$\frac{1}{2}\left(e^{2}-1\right)=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^{2}+\frac{1}{2}\frac{\ell^{2}}{r^{2}}-GM\left(\frac{1}{r}+\frac{\ell^{2}}{r^{3}}\right) \tag{23}$$

$$\frac{dr}{d\tau}=\pm\left[e^{2}-1+2GM\left(\frac{1}{r}+\frac{\ell^{2}}{r^{3}}\right)-\frac{\ell^{2}}{r^{2}}\right]^{1/2} \tag{24}$$

$$=\pm\sqrt{e^{2}-\left(1-\frac{2GM}{r}\right)\left(1+\frac{\ell^{2}}{r^{2}}\right)} \tag{25}$$

Since the only dependent variable in this equation is r (ℓ and e are constants), this is a first-order ODE for r in terms of the proper time τ.

We can write 25 in the form giving the radial equation of motion:

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^{2}+\frac{1}{2}\frac{l^{2}}{r^{2}}-GM\left(\frac{1}{r}+\frac{l^{2}}{r^{3}}\right)=\frac{1}{2}\left(e^{2}-1\right)} \tag{26}$$

Example 1. Suppose we start a particle at rest at infinity and let it fall in radially towards the mass. Since the motion is entirely radial, the angular momentum is $\ell=0$. Also, since the particle starts off at rest $e=1$ (see above), so the equation becomes

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^{2}=\frac{GM}{r} \tag{27}$$

$$\pm\int\sqrt{r}dr=\sqrt{2GM}\int d\tau \tag{28}$$

$$\pm\frac{2}{3}r^{3/2}=\sqrt{2GM}\tau+T \tag{29}$$

where T is a constant of integration. If the particle is falling inwards, then we would expect τ to increase as r decreases, so we need to take the minus sign on the left. The proper time interval as measured by the falling object between two radii (say, $r_A=10GM$ and $r_B=2GM$) is

$$\Delta\tau = \frac{2}{3\sqrt{2GM}}\left((10GM)^{3/2} - (2GM)^{3/2}\right) \tag{30}$$

$$= \frac{\sqrt{2}}{3}\left(10^{3/2} - 2^{3/2}\right)GM \tag{31}$$

PINGBACKS

Pingback: Circular orbits - Kepler's law
Pingback: Circular orbits - relation between radius and angular momentum
Pingback: Particles falling towards a mass
Pingback: Particle falling towards a mass - two types of velocity
Pingback: Vertical particle motion
Pingback: Orbit of a comet around a black hole
Pingback: Twin paradox with a black hole
Pingback: Perihelion shift in planetary orbits
Pingback: Perihelion shift - contribution of the time coordinate
Pingback: Perihelion shift - numerical solution
Pingback: Photon equations of motion
Pingback: Local flat coordinate systems - four-momentum of photons
Pingback: Apparent size of a black hole to a moving observer
Pingback: Circular orbit - appearance to a falling observer
Pingback: Local flat frame for a circular orbit
Pingback: Proper time to fall through the event horizon
Pingback: Falling object observed near the event horizon
Pingback: Escape velocity near an event horizon
Pingback: Painlevé-Gullstrand (global rain) coordinates
Pingback: Black hole radiation - energy of emitted particles
Pingback: Black hole radiation - energy at infinity of radiated particle
Pingback: Black hole heat engine
Pingback: Schwarzschild metric with negative mass
Pingback: Schwarzschild metric with non-zero cosmological constant