PENROSE DIAGRAMS IN FLAT SPACETIME

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An alternative to the traditional spacetime diagram in special relativity is the *Penrose diagram*. Penrose diagrams are also applicable to general relativity, although there they get a bit more complex, so we'll look at them in flat spacetime here.

A Penrose diagram is obtained from the usual spacetime coordinates (in 2 dimensions) t and r (here, we're using polar coordinates). In ordinary polar coordinates, the interval is, for radial motion

$$ds^2 = -dt^2 + dr^2 \tag{1}$$

We consider the transformation

$$\begin{aligned} u &\equiv t - r \\ v &\equiv t + r \end{aligned} \tag{2}$$

and its inverse

$$\begin{aligned} t &= \frac{1}{2}(u+v) \\ r &= \frac{1}{2}(v-u) \end{aligned} \tag{3}$$

Substituting into 1 we have

$$ds^2 = -\frac{1}{4}(du+dv)^2 + \frac{1}{4}(dv-du)^2 \tag{4}$$

$$= -du\,dv \tag{5}$$

A light ray has $t = \pm r + k$ (for constant k) so that $ds^2 = 0$. From this we see that either $u = k$ or $v = k$ along the path of a light ray, so lines of constant u and v correspond to light trajectories.

To get to a Penrose diagram, we make a further transformation to coordinates t' and r', defined by

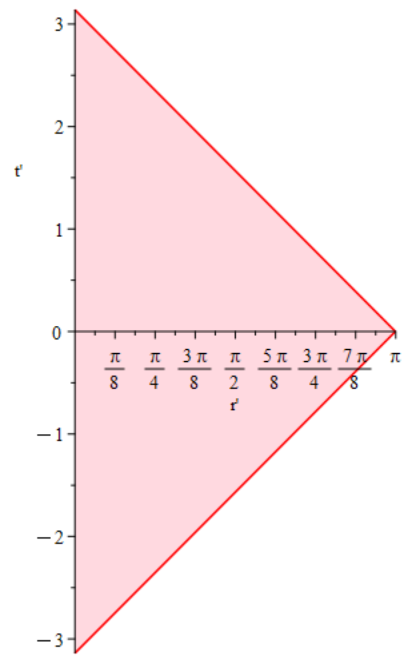

FIGURE 1. Penrose diagram for flat spacetime.

$$\begin{aligned} \tan^{-1} u &\equiv \frac{1}{2}\left(t' - r'\right) \\ \tan^{-1} v &\equiv \frac{1}{2}\left(t' + r'\right) \end{aligned} \tag{6}$$

Inverting this we get

$$\begin{aligned} t' &= \tan^{-1}(t-r) + \tan^{-1}(t+r) \\ r' &= -\tan^{-1}(t-r) + \tan^{-1}(t+r) \end{aligned} \tag{7}$$

If we restrict $\tan^{-1}$ to its principal branch so that it has values in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then t' and r' have finite ranges, even though t ranges from $-\infty$ to $+\infty$ and r ranges from 0 to ∞. If $t \to \infty$ and $r = 0$, $t' \to \pi$ and $r' \to 0$; if $t \to -\infty$ and $r = 0$, then $t' \to -\pi$ and $r' \to 0$; and if $t = 0$ and $r \to \infty$, then $t' \to 0$ and $r' \to \pi$. Thus the extremes in t and r map into the corners of a triangle, as shown in Fig. 1. Thus the infinite ranges of t and r all map into the shaded region in Fig. 1.

From 6, we see that one boundary lies at $\tan^{-1} u = -\frac{\pi}{2}$, which is the line $t' = r' - \pi$ (the lower side to the triangle in Fig. 1). Another boundary is $\tan^{-1} v = \frac{\pi}{2}$, corresponding to the line $t' = -r' + \pi$, which is the upper side

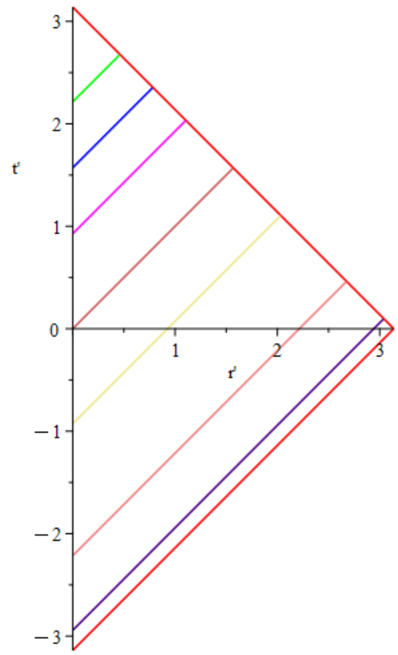

FIGURE 2. Light rays following $t = r + k$ for various k.

to the triangle. [We could also define a boundary by taking $\tan^{-1} u = +\frac{\pi}{2}$, but this would give us a line starting at $t' = \pi$ and $r' = 0$ and going upwards from the top vertex in Fig. 1. This would be the result of extending the arctan outside its principal branch. A similar argument could be made for a boundary defined by $\tan^{-1} v = -\frac{\pi}{2}$, which would start a boundary at the lower vertex and extend downwards.]

The paths of light rays can be plotted on a Penrose diagram by setting $t = \pm r + k$ for various constants k. In Fig. 2 we see the paths of light rays for 7 values of k.

From the bottom up, we have $k = -10$ (indigo), -2 (orange), -0.5 (khaki), 0 (brown), +0.5 (magenta), 1 (blue) and 2 (green). The paths start on the t' axis at $r = 0$ and proceed towards the upper right as r (and therefore also r') increases.

We can do a similar plot for light rays travelling inwards, towards decreasing r, as in Fig. 3. These plots were obtained from the condition $t = -r + k$, with $k = -10$ (indigo), -2 (orange), -0.5 (khaki), 0 (brown), +0.5 (magenta), 1 (blue) and 2 (green).

The world lines for light rays are straight lines at $\pm 45°$, just as in ordinary spacetime diagrams, so timelike paths (that massive particles can follow) lie within the light cones defined by these lines, just as before.

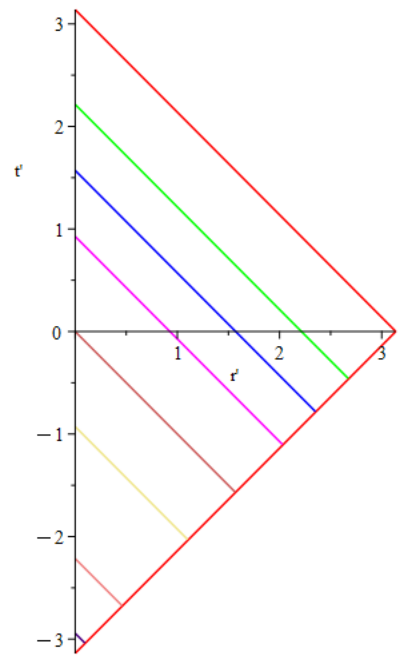

FIGURE 3. Light rays following $t = -r + k$ for various k.

As an example, Fig. 4 shows 3 world lines with constant r. The brown curve is for $r = 0.5$, blue for $r = 1$ and green for $r = 2$. The time runs from -100 to $+100$ (which is effectively infinite at the scale of the diagram). The tangents to all three curves lie within the light cone boundaries.

We can also look at spacelike world lines by holding t constant and varying r as in Fig. 5. The violet curve is with $t = -2$. Proceeding upwards we have $t = -1$ (khaki), $t = -0.5$ (olive), $t = 0.5$ (brown), $t = 1$ (blue) and $t = 2$ (green). The tangents to all these curves lie outside the light cone.

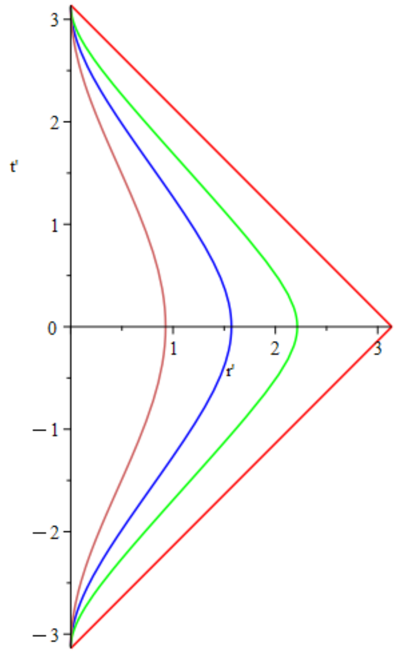

FIGURE 4. World lines with constant r.

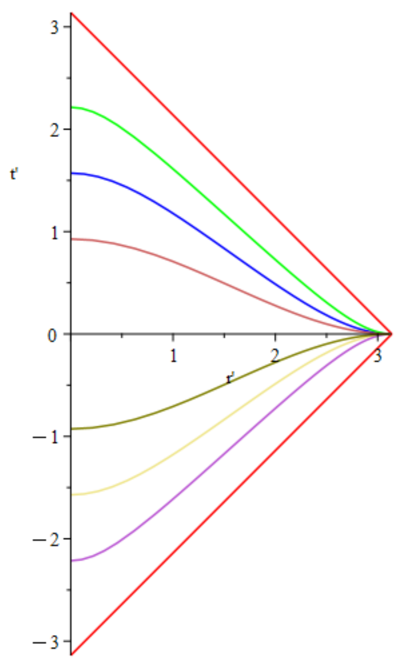

FIGURE 5. World lines with constant t.