PERIHELION SHIFT - CONTRIBUTION OF THE TIME COORDINATE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Post date: 8 Sep 2022.

We've seen that a third of the precession of an object's closest approach in its orbit around a central mass comes from the curvature of space caused by the radial component of the Schwarzschild metric. We can now investigate the contribution from the time coordinate. For nearly circular orbits where the mean radius r_c of the orbit is much greater than GM (M is the central mass), the angle of closest approach advances by

$$\Delta\phi = \frac{6\pi GM}{r_c} \tag{1}$$

on each orbit.

By analogy with the radial coordinate, we might think that we can analyze the time component by setting $dr = 0$. However, since we're now concerned with changes in the way time is perceived at different locations, rather than merely with the curvature of space at a particular time, we have to consider the change in position with time, so our approach has to be quite different from that for the radial coordinate. The analysis is, in fact, much the same as in the original derivation of the full precession using the Schwarzschild metric. Since we're interested in looking only at the time coordinate, we consider a modified metric in which space is flat, which means that the metric component $g_{rr} = 1$ rather than the $\left(1 - \frac{2GM}{r}\right)^{-1}$ in the Schwarzschild metric. That is, the metric is (since we're still considering motion in the equatorial plane so $d\theta = 0$):

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + dr^2 + r^2 d\phi^2 \tag{2}$$

Since the t and ϕ components are unchanged from the original metric, the conserved quantities e and ℓ are also unchanged:

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{3}$$

$$\ell = r^2 \frac{d\phi}{d\tau} \tag{4}$$

Using the invariant $\mathbf{u}\cdot\mathbf{u}=-1$ we get

$$-1=-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^{2}+\left(\frac{dr}{d\tau}\right)^{2}+r^{2}\left(\frac{d\phi}{d\tau}\right)^{2} \tag{5}$$

$$1=\left(1-\frac{2GM}{r}\right)^{-1}e^{2}-\left(\frac{dr}{d\tau}\right)^{2}-\frac{\ell^{2}}{r^{2}} \tag{6}$$

Now we use

$$\frac{dr}{d\tau}=\frac{dr}{d\phi}\frac{d\phi}{d\tau}=\frac{\ell}{r^{2}}\frac{dr}{d\phi} \tag{7}$$

$$1=\left(1-\frac{2GM}{r}\right)^{-1}e^{2}-\frac{\ell^{2}}{r^{4}}\left(\frac{dr}{d\phi}\right)^{2}-\frac{\ell^{2}}{r^{2}} \tag{8}$$

Substituting $u=1/r$, we get

$$1=(1-2GMu)^{-1}e^{2}-\ell^{2}\left(\frac{du}{d\phi}\right)^{2}-\ell^{2}u^{2} \tag{9}$$

Taking the derivative with respect to ϕ and using the notation u' to denote a derivative, we get

$$2GM(1-2GMu)^{-2}e^{2}u'-2\ell^{2}u'u''-2\ell^{2}uu'=0 \tag{10}$$

$$u''+u=\frac{GMe^{2}}{\ell^{2}(1-2GMu)^{2}} \tag{11}$$

If we now assume that $r\gg GM$ as usual so that u is small, we can approximate the RHS term:

$$u''+u\approx\frac{GMe^{2}}{\ell^{2}}(1+4GMu) \tag{12}$$

$$u''+u\left(1-4\left(\frac{GMe}{\ell}\right)^{2}\right)=\frac{GMe^{2}}{\ell^{2}} \tag{13}$$

Now we want to treat a nearly circular orbit as a purely circular orbit plus a small perturbation, as we did in the original perihelion calculation. If $u=u_c=$ constant, then the above equation becomes

$$u_c\left(1-4\left(\frac{GMe}{\ell}\right)^2\right)=\frac{GMe^2}{\ell^2} \tag{14}$$

$$=\frac{1}{GM}\left(\frac{GMe}{\ell}\right)^2 \tag{15}$$

$$\left(\frac{GMe}{\ell}\right)^2=\frac{GMu_c}{4GMu_c+1} \tag{16}$$

Substituting this into the ODE, we get

$$u''+u\left(1-\frac{GMu_c}{4GMu_c+1}\right)=\frac{u_c}{4GMu_c+1} \tag{17}$$

Now we define $u=u_c+u_cw$, where w is the small perturbation on the circular orbit. Plugging this into the differential equation above, we get, after cancelling u_c off both sides:

$$w''+(1+w)\frac{1}{4GMu_c+1}=\frac{1}{4GMu_c+1} \tag{18}$$

$$w''=-\frac{1}{4GMu_c+1}w \tag{19}$$

This is the familiar harmonic oscillator equation again, so we can write the solution in which $\phi=0$ is the maximum value of w:

$$w(\phi)=A\cos\left(\frac{\phi}{\sqrt{4GMu_c+1}}\right) \tag{20}$$

A complete orbit will bring the argument of the cosine to 2π, so

$$\phi=2\pi\sqrt{4GMu_c+1} \tag{21}$$

$$\approx 2\pi(1+2GMu_c) \tag{22}$$

The perihelion shift due to the t coordinate is then

$$\Delta\phi=4\pi GMu_c=\frac{4\pi GM}{r_c} \tag{23}$$

which is $\frac{2}{3}$ of the total amount stated at the start. Thus the radial plus time coordinates together account for the total perihelion shift (in the approximation of large radii and nearly circular orbits, anyway).