PERIHELION SHIFT IN PLANETARY ORBITS

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If a planet is in an elliptical orbit around the sun (as they all are), there is one point in the orbit where the planet approaches most closely to the sun; this point is called the *perihelion*. In an ideal two-body system (that is, a system composed of the sun and a single planet) in Newtonian mechanics, the perihelion occurs at the same angular location from one orbit to the next. For example, if the planet's perihelion occurred at $\phi = 0$ on one orbit, it will always occur at $\phi = 0$ on all future orbits.

Newtonian gravitational theory predicts that the presence of other planets will cause a perihelion shift, so that the angular location of the closest approach advances by a small angle on each orbit. By the time Einstein proposed relativity theory, the measurements of the perihelion angles of the various planets had become quite accurate, and the calculations from Newtonian theory had also reached a high degree of precision. This gave rise to one of the big problems of early 20th century physics: the perihelion shift of Mercury was about 43 seconds of arc per century larger than Newtonian theory said it should be. Now this may not sound like much, but the measurements and calculations were sound enough that this discrepancy was undoubtedly real, and not the result of observational or theoretical error.

One of the early triumphs of general relativity was its ability to explain this discrepancy, which we'll have a look at here. In the general multi-planet system, we must use computer simulation to solve the equations, but a relatively simple calculation can be done in the case of a single planet orbiting the sun, if we assume the orbit is roughly circular (as all planetary orbits are) and that the average radius of the orbit is much greater than GM for the sun (since $GM = 1.477$ km, this is certainly true of all planets).

The main conceptual difference between the predictions of Newtonian and relativistic theories is that in Newtonian gravity, it is the perturbations due to the other planets that cause the perihelion shift, while in relativity, the shift is due to the curvature of spacetime, and is present even in a two-body system without perturbations due to other planets. Perturbations due to the other planets also have an effect in relativity, but there is an additional shift due to the Schwarzschild metric, which we'll estimate here.

We would like to find an expression for r as a function of ϕ. We start with the conserved quantities (remember we're taking the orbit to be in the plane $\theta = \frac{\pi}{2}$):

$$\tilde{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\ell^2}{r^2} - GM\left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) \tag{1}$$

$$\ell = r^2\frac{d\phi}{d\tau} \tag{2}$$

Using the chain rule, we can write

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\frac{\ell}{r^2} \tag{3}$$

Substituting this into the energy equation, we get

$$\tilde{E} = \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2\frac{\ell^2}{r^4} + \frac{1}{2}\frac{\ell^2}{r^2} - GM\left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) \tag{4}$$

If we are given $\tilde{E}$ and ℓ, then this is an ODE for $r(\phi)$. As you might imagine, it's impossible to integrate analytically, but we can apply the assumptions above to get a more pleasant equation. First, we define $u \equiv 1/r$ so that

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \tag{5}$$

$$\frac{dr}{d\phi} = -r^2\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi} \tag{6}$$

Therefore

$$\tilde{E} = \frac{\ell^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{\ell^2}{2}u^2 - GM\left(u + \ell^2u^3\right) \tag{7}$$

We can now take the derivative of this equation with respect to ϕ (and we'll use the notation $u' = du/d\phi$ and so on):

$$\ell^2u'u'' + \ell^2uu' - GM\left(u' + 3\ell^2u^2u'\right) = 0 \tag{8}$$

We can cancel off a factor of u' since $u' = -\frac{1}{r^2}r' = 0$ would imply that $r' = 0$, in other words, a circular orbit, which we're not interested in here. Thus we get

$$u'' + u = \frac{GM}{\ell^2} + 3GMu^2 \tag{9}$$

We'll assume that the orbit is nearly circular, which we can formalize by saying

$$u = u_c + u_c w(\phi) \tag{10}$$

where u_c is a constant (and can be taken as $u_c = 1/r_c$, where r_c is the mean radius), and w is a perturbation function, which is assumed to be small for all values of ϕ. (We don't need to multiply w by u_c, but it makes the calculation a bit cleaner.)

We now substitute this into the ODE above:

$$u_c w'' + u_c + u_c w = \frac{GM}{\ell^2} + 3GM(u_c + u_c w)^2 \tag{11}$$

For a circular orbit, there is a relation between radius and angular momentum

$$\ell^2 = \frac{r_c^2 GM}{r_c - 3GM} \tag{12}$$

$$= \frac{GM}{u_c(1 - 3GMu_c)} \tag{13}$$

Substituting this, we get

$$u_c w'' + u_c + u_c w = u_c(1 - 3GMu_c) + 3GM(u_c + u_c w)^2 \tag{14}$$

$$u_c w'' + u_c w = 6GMu_c^2 w + 3GMu_c^2 w^2 \tag{15}$$

$$w'' = (6GMu_c - 1)w + 3GMu_c^2 w^2 \tag{16}$$

Now for the first approximation: since we're assuming nearly circular orbits, we can take w to be very small, so we can neglect the quadratic term. We then get

$$w'' \approx -(1 - 6GMu_c)w \tag{17}$$

This is the classic equation for a harmonic oscillator and has solution

$$w(\phi) = A\cos\left(\sqrt{1 - 6GMu_c}\phi\right) + B\sin\left(\sqrt{1 - 6GMu_c}\phi\right) \tag{18}$$

Since $u = 1/r$, the maximum value of w corresponds to the minimum distance (the perihelion) in the orbit. If we choose the ϕ coordinate so that the first maximum occurs at $\phi = 0$, then $B = 0$. The next maximum will occur when

$$\phi = \frac{2\pi}{\sqrt{1-6GMu_c}} \tag{19}$$

Now we can apply the second approximation, which is that $r_c \gg GM$, or $GMu_c \ll 1$. We can expand the square root in a Taylor series up to first order, so that

$$\phi = 2\pi\left(1+\frac{3GM}{r_c}+\mathcal{O}\left(\frac{1}{r_c^2}\right)\right) \tag{20}$$

To first order, the second perihelion occurs at

$$\phi = 2\pi + \frac{6\pi GM}{r_c} \tag{21}$$

Thus the predicted perihelion shift per orbit is $\frac{6\pi GM}{r_c}$.

How does this compare with the observed value for Mercury? The required values are

$$GM = 1.477 \text{ km} \tag{22}$$

$$r_c = 5.79\times 10^7 \text{ km} \tag{23}$$

$$T = 0.241 \text{ years} \tag{24}$$

To convert the result into arc seconds per century, we need to multiply as follows

$$\Delta\phi = \frac{6\pi GM}{r_c}\left(\frac{180}{\pi}\text{deg rad}^{-1}\right)\left(3600 \text{ arcsec deg}^{-1}\right)\left(\frac{100}{0.241}\text{orbits century}^{-1}\right) \tag{25}$$

$$= 41.15 \text{ arcsec century}^{-1} \tag{26}$$

The agreement is quite good, even with this approximate solution. Remember that this is the perihelion shift due exclusively to the curvature of spacetime, and is in addition to the shift due to the perturbations due to the other planets. In other words, the curvature of spacetime accounts for the discrepancy between observation and Newtonian theory.

We can work out the corresponding perihelion shifts for Venus, Earth and Mars. We need to substitute in the corresponding values of r_c and T.

For Venus, $r_c = 1.082\times 10^8$ km and $T = 0.615$ years giving $\Delta\phi = 8.63$ arcsec century^{-1}.

For Earth, $r_c = 1.496\times 10^8$ km and $T = 1$ year giving $\Delta\phi = 3.84$ arcsec century^{-1}.

For Mars, $r_c = 2.279\times 10^8$ km and $T = 1.881$ years giving $\Delta\phi = 1.34$ arcsec century^{-1}.

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