PHOTON EQUATIONS OF MOTION

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The main problem in applying the geodesic equation to photons is that for photons, $d\tau$ is always zero (that is, the proper time in the frame of the photon never changes). The geodesic equation makes explicit reference to τ:

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{1}$$

The resolution of this problem is typical of the approach to relativity: find some equations for ordinary particles (that is, particles with rest mass) that *don't* depend on τ or m (since photons have no rest mass, we can't have mass showing up in the equations either), and then *assume* that these equations are valid for photons too.

We can start with the conserved quantities in the Schwarzschild metric:

$$e=\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{2}$$

$$\ell=r^2\sin^2\theta\frac{d\phi}{d\tau} \tag{3}$$

These quantities both make reference to τ but their *ratio* doesn't. If we further assume that motion is in the equatorial plane so that $\theta=\pi/2$, then we get

$$\frac{\ell}{e}=\frac{r^2\frac{d\phi}{d\tau}}{\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}} \tag{4}$$

$$=\left(1-\frac{2GM}{r}\right)^{-1}r^2\frac{d\phi}{dt} \tag{5}$$

where t is now the Schwarzschild time coordinate. Thus one equation of motion is

$$\frac{d\phi}{dt}=\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)b \tag{6}$$

where $b\equiv\ell/e$.

A second equation can be derived starting from the Schwarzschild metric definition:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{7}$$

For a photon, $ds^2 = 0$, and since we're restricting motion to the equatorial plane $d\theta = 0$, so

$$-\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\phi^2 = 0 \tag{8}$$

Dividing through by $\left(1 - \frac{2GM}{r}\right) dt^2$ and using 6, we get

$$\left[\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt}\right]^2 + \left(1 - \frac{2GM}{r}\right)^{-1} r^2 \left(\frac{d\phi}{dt}\right)^2 = 1 \tag{9}$$

$$\left[\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt}\right]^2 + \left(1 - \frac{2GM}{r}\right) \frac{b^2}{r^2} = 1 \tag{10}$$

$$\left[\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt}\right]^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{1}{b^2} \tag{11}$$

This last equation is the equation of motion for r.

If we interpret this as a sort of kinetic energy plus potential energy equation, we can define the pseudo-kinetic energy as

$$K \equiv \left[\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt}\right]^2 \tag{12}$$

and the pseudo-potential energy as

$$U(r) \equiv \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{13}$$

The potential energy has a zero at $r = 2GM$ and a maximum found from its derivative

$$\frac{dU}{dr} = -\frac{2}{r^3} + \frac{6GM}{r^4} = 0 \tag{14}$$
$$r = 3GM \tag{15}$$
$$U_{max} = \frac{1}{27G^2M^2} \tag{16}$$

The ratio $\ell/e = b$ must be specified in order for the equations of motion to be solved. We can put a geometric interpretation on b, which helps us to understand what the equations of motion are saying. First, draw a ray from $r = 0$ out to infinity, and suppose a photon starts at infinity and follows an initial path that is parallel to this ray, but at a perpendicular distance d from it. Now draw a triangle with one side being the line from $r = 0$ to the photon, a second side being along the ray we drew at the start, and the third side being the perpendicular line of length d that goes from the photon's path to the ray. For large r, the first two sides of the triangle will be almost the same length, and the angle ϕ subtended by the side d will be very small, so that $\sin\phi \approx \phi \approx d/r$. Taking the derivative, we get

$$\frac{d\phi}{dt} = -\frac{d}{r^2}\frac{dr}{dt} \tag{17}$$

From the equation of motion for r, at very large r we get, by ignoring terms where r has a negative exponent:

$$\frac{1}{b^2}\left(\frac{dr}{dt}\right)^2 \approx \frac{1}{b^2} \tag{18}$$
$$\frac{dr}{dt} \approx \pm 1 \tag{19}$$

This is consistent, since at very large r, space is essentially flat, so the speed of the photon is just dr/dt which is the speed of light. Putting this into 17, we get (taking $dr/dt = -1$ for an incoming photon):

$$\frac{d\phi}{dt} = -\frac{d}{r^2}\frac{dr}{dt} = \frac{d}{r^2} \tag{20}$$

However, from 6 at very large r $d\phi/dt \approx b/r^2$, so it seems that $b = d$, that is, b is what is known as the *impact parameter.* In flat space, b is the distance of closest approach to $r = 0$.

Returning to the Schwarzschild metric, in the equation of motion 11 for r above, we see that if $b^2 < 27G^2M^2$ then dr/dt can never be zero, since U_{max} in 16 is always less than $1/b^2$. Thus if the impact parameter is $b < \sqrt{27}GM$ and $dr/dt < 0$ initially (that is, the photon is approaching $r = 0$), then it must continue to approach $r=0$ forever. The only way it can do this is to spiral in towards the origin.

If $b>\sqrt{27}GM$, there will be two positive values of r for which $U(r)=1/b^2$ (since this equation is actually a cubic equation in r, there are 3 solutions, but one of them is always negative, which we ignore). The values of r between these solutions make $U>1/b^2$ which is not allowed, since that would require the first term, which is a square, to be negative. Thus a photon coming in with $dr/dt<0$ approaches until r is the higher of the two solutions, at which point $dr/dt=0$. After that, dr/dt becomes positive, and the photon recedes in some other direction.

For example, if $b=6GM$, the photon will approach until it reaches the larger root, which is $r=4.453GM$ and then recede to infinity.

If we consider a cylinder of light heading in towards the central mass, then all photons with $b<\sqrt{27}GM$ will be absorbed by the mass, thus all light within a cylinder of radius $R=\sqrt{27}GM$ will be absorbed. Photons with impact parameters outside this value will not be absorbed, but will be deflected by the mass and continue on back out to infinity.

If $b=\sqrt{27}GM$, then there is only one positive value of r for which $U=1/b^2$, and that is $r=3GM$. When the photon reaches that radius, it can continue either by spiralling inwards or receding to infinity, or by entering an unstable circular orbit.

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