PLANE SYMMETRIC SPACETIME

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A static, plane-symmetric spacetime is one in which spacetime is independent of time (static) and is composed of a set of planes, where each plane is labelled by a coordinate x. Within each plane, points are labelled by coordinates y and z and because the spacetime is static, the distance between two points depends only on these two coordinates:

$$\left[ds^2\right]_x = dy^2 + dz^2 \tag{1}$$

where the subscript x denotes the plane with coordinate x.

If the x basis vector $\mathbf{e}_x$ is everywhere perpendicular to $\mathbf{e}_y$ and $\mathbf{e}_z$ (and $\mathbf{e}_y \perp \mathbf{e}_z$), then the spatial off-diagonal components of the metric are zero

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j \tag{2}$$

$$g_{xy} = g_{xz} = g_{yz} = 0 \tag{3}$$

The general metric between any two spacetime points is then

$$ds^2 = g_{tt}dt^2 + 2g_{tx}dt\,dx + dx^2 + dy^2 + dz^2 \tag{4}$$

Because the spacetime is static, a displacement forward in time by dt should give the same separation as a displacement backwards by the same amount $-dt$. Because of this symmetry, the $2g_{tx}dt\,dx$ term should remain unchanged when dt is replaced by $-dt$. However, since the metric is independent of time, $g_{tx}(t) = g_{tx}(-t)$, so the only way we can satisfy the symmetry requirement is if $g_{tx} = 0$. Thus the plane-symmetric metric is symmetric:

$$ds^2 = g_{tt}dt^2 + dx^2 + dy^2 + dz^2 \tag{5}$$

Further, g_{tt} can depend at most on x alone.

To work out the consequences of this metric, we need to evaluate the Christoffel symbols and Ricci tensor. The Christoffel symbol worksheet is:

$\Gamma_{00}^{0}=\frac{1}{2A}A_{0}$	$\Gamma_{10}^{0}=\Gamma_{01}^{0}=\frac{1}{2A}A_{1}$	$\Gamma_{20}^{0}=\Gamma_{02}^{0}=\frac{1}{2A}A_{2}$	$\Gamma_{30}^{0}=\Gamma_{03}^{0}=\frac{1}{2A}A_{3}$
$\Gamma_{11}^{0}=\frac{1}{2A}B_{0}$	$\Gamma_{22}^{0}=\frac{1}{2A}C_{0}$	$\Gamma_{33}^{0}=\frac{1}{2A}D_{0}$	other $\Gamma_{\mu\nu}^{0}=0$
$\Gamma_{01}^{1}=\Gamma_{10}^{1}=\frac{1}{2B}B_{0}$	$\Gamma_{11}^{1}=\frac{1}{2B}B_{1}$	$\Gamma_{12}^{1}=\Gamma_{21}^{1}=\frac{1}{2B}B_{2}$	$\Gamma_{13}^{1}=\Gamma_{31}^{1}=\frac{1}{2B}B_{3}$
$\Gamma_{00}^{1}=\frac{1}{2B}A_{1}$	$\Gamma_{22}^{1}=-\frac{1}{2B}C_{1}$	$\Gamma_{33}^{1}=-\frac{1}{2B}D_{1}$	other $\Gamma_{\mu\nu}^{1}=0$
$\Gamma_{02}^{2}=\Gamma_{20}^{2}=\frac{1}{2C}C_{0}$	$\Gamma_{12}^{2}=\Gamma_{21}^{2}=\frac{1}{2C}C_{1}$	$\Gamma_{22}^{2}=\frac{1}{2C}C_{2}$	$\Gamma_{32}^{2}=\Gamma_{23}^{2}=\frac{1}{2C}C_{3}$
$\Gamma_{00}^{2}=\frac{1}{2C}A_{2}$	$\Gamma_{11}^{2}=-\frac{1}{2C}B_{2}$	$\Gamma_{33}^{2}=-\frac{1}{2C}D_{2}$	other $\Gamma_{\mu\nu}^{2}=0$
$\Gamma_{03}^{3}=\Gamma_{30}^{3}=\frac{1}{2D}D_{0}$	$\Gamma_{13}^{3}=\Gamma_{31}^{3}=\frac{1}{2D}D_{1}$	$\Gamma_{23}^{3}=\Gamma_{32}^{3}=\frac{1}{2D}D_{2}$	$\Gamma_{33}^{3}=\frac{1}{2D}D_{3}$
$\Gamma_{00}^{3}=\frac{1}{2D}A_{3}$	$\Gamma_{11}^{3}=-\frac{1}{2D}B_{3}$	$\Gamma_{22}^{3}=-\frac{1}{2D}C_{3}$	other $\Gamma_{\mu\nu}^{3}=0$

In this case $\left(x^{0},x^{1},x^{2},x^{3}\right)=(t,x,y,z)$ and

$$A=-g_{tt}(x) \tag{6}$$
$$B=1 \tag{7}$$
$$C=1 \tag{8}$$
$$D=1 \tag{9}$$

Thus the only nonzero symbols will be those involving A_{1}, since all other derivatives are zero. These are

$$\Gamma_{10}^{0}=\Gamma_{01}^{0}=\frac{1}{2A}A_{1}=\frac{1}{2A}\frac{dA}{dx} \tag{10}$$
$$\Gamma_{00}^{1}=\frac{1}{2B}A_{1}=\frac{1}{2}\frac{dA}{dx} \tag{11}$$

[We can use the total derivative rather than partial because A depends only on x.]

From the Ricci tensor worksheet, the only nonzero components of $R_{\mu\nu}$ are those involving A_{11} or A_{1} only, so we see that

$$R_{00} = \frac{1}{2B}A_{11} - \frac{1}{4BA}A_1^2 \tag{12}$$

$$= \frac{1}{2}\frac{d^2A}{dx^2} - \frac{1}{4A}\left(\frac{dA}{dx}\right)^2 \tag{13}$$

$$R_{11} = -\frac{1}{2}\frac{d^2A}{dx^2} + \frac{1}{4A}\left(\frac{dA}{dx}\right)^2 \tag{14}$$

with all other $R_{\mu\nu} = 0$. In flat space, all components satisfy $R_{\mu\nu} = 0$ so these two components both give the same condition on A:

$$\frac{d^2A}{dx^2} = \frac{1}{2A}\left(\frac{dA}{dx}\right)^2 \tag{15}$$

To examine the structure of the spacetime, we need the full Riemann tensor, which is defined in terms of the Christoffel symbols:

$$R_{\epsilon\nu\lambda\sigma} = g_{\epsilon\mu}R^{\mu}_{\ \nu\lambda\sigma} = g_{\epsilon\mu}\left[-\partial_\sigma\Gamma^{\mu}_{\ \lambda\nu} + \partial_\lambda\Gamma^{\mu}_{\ \sigma\nu} - \Gamma^{\kappa}_{\ \lambda\nu}\Gamma^{\mu}_{\ \kappa\sigma} + \Gamma^{\kappa}_{\ \sigma\nu}\Gamma^{\mu}_{\ \lambda\kappa}\right] \tag{16}$$

We can work out the terms in $R^{\mu}_{\ \nu\lambda\sigma}$ using 10 and 11. First, we'll expand the implied sums and label the terms:

$$-\Gamma^{\kappa}_{\ \lambda\nu}\Gamma^{\mu}_{\ \kappa\sigma} = \overbrace{-\Gamma^{0}_{\ \lambda\nu}\Gamma^{\mu}_{\ 0\sigma}}^{[1]}\overbrace{-\Gamma^{1}_{\ \lambda\nu}\Gamma^{\mu}_{\ 1\sigma}}^{[2]} \tag{17}$$

$$\Gamma^{\kappa}_{\ \sigma\nu}\Gamma^{\mu}_{\ \lambda\kappa} = \overbrace{\Gamma^{0}_{\ \sigma\nu}\Gamma^{\mu}_{\ \lambda 0}}^{[3]}\overbrace{+\Gamma^{1}_{\ \sigma\nu}\Gamma^{\mu}_{\ \lambda 1}}^{[4]} \tag{18}$$

$$-\partial_\sigma\Gamma^{\mu}_{\ \lambda\nu} + \partial_\lambda\Gamma^{\mu}_{\ \sigma\nu} = \overbrace{-\partial_\sigma\Gamma^{\mu}_{\ \lambda\nu}}^{[5]}\overbrace{+\partial_\lambda\Gamma^{\mu}_{\ \sigma\nu}}^{[6]} \tag{19}$$

Next, we'll identify the index combinations that give (potentially) nonzero values for components of $R^{\mu}_{\ \nu\lambda\sigma}$ in each term, using the fact that only Γ^{0}_{10} and Γ^{1}_{00} are nonzero, and that only the derivative with respect to x (index 1) is nonzero.

- Term 1:

μ	ν	λ	σ
1	1	0	0
1	0	1	0
0	1	0	1
0	0	1	1

- Term 2:

μ	ν	λ	σ
0	0	0	0

- Term 3:

μ	ν	λ	σ
0	0	1	1
1	0	0	1
0	1	1	0
1	1	0	0

- Term 4:

μ	ν	λ	σ
0	0	0	0

- Term 5:

μ	ν	λ	σ
0	0	1	1
0	1	0	1
1	0	0	1

- Term 6:

μ	ν	λ	σ
0	0	1	1
0	1	1	0
1	0	1	0

From these tables, we see that there are 7 unique index combinations that can potentially give nonzero Riemann tensor components $R^{\mu}_{\ \nu\lambda\sigma}$. We have (remember that the Christoffel symbols are symmetric in their lower 2 indices: $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$):

$$R^{1}_{\ 100} = -\Gamma^{0}_{01}\Gamma^{1}_{00} + \Gamma^{0}_{01}\Gamma^{1}_{00} = 0 \tag{20}$$
$$R^{0}_{\ 011} = -\Gamma^{0}_{01}\Gamma^{0}_{01} + \Gamma^{0}_{01}\Gamma^{0}_{01} - \partial_1\Gamma^{0}_{01} + \partial_1\Gamma^{0}_{01} = 0 \tag{21}$$
$$R^{0}_{\ 000} = -\Gamma^{1}_{00}\Gamma^{0}_{10} + \Gamma^{1}_{00}\Gamma^{0}_{10} = 0 \tag{22}$$
$$R^{1}_{\ 010} = -\Gamma^{0}_{01}\Gamma^{1}_{00} + \partial_1\Gamma^{1}_{00} \tag{23}$$
$$R^{1}_{\ 001} = +\Gamma^{0}_{01}\Gamma^{1}_{00} - \partial_1\Gamma^{1}_{00} = -R^{1}_{\ 010} \tag{24}$$
$$R^{0}_{\ 101} = -\Gamma^{0}_{01}\Gamma^{0}_{01} - \partial_1\Gamma^{0}_{01} \tag{25}$$
$$R^{0}_{\ 110} = \Gamma^{0}_{01}\Gamma^{0}_{01} + \partial_1\Gamma^{0}_{01} = -R^{0}_{\ 101} \tag{26}$$

Thus only the last 4 can potentially be nonzero. To go further, we need the derivative terms:

$$\partial_x\Gamma^{0}_{\ 10} = -\frac{1}{2A^2}\left(\frac{dA}{dx}\right)^2 + \frac{1}{2A}\frac{d^2A}{dx^2} \tag{27}$$
$$= -\frac{1}{2A^2}A_1^2 + \frac{1}{2A}A_{11} \tag{28}$$
$$\partial_x\Gamma^{1}_{\ 00} = \frac{1}{2}\frac{d^2A}{dx^2} = \frac{1}{2}A_{11} \tag{29}$$

Now we can use 10 and 11 to write these components in terms of A:

$$R^{1}{}_{010} = -\Gamma^{0}_{01}\Gamma^{1}_{00} + \partial_1\Gamma^{1}_{00} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \tag{30}$$

$$R^{1}{}_{001} = -R^{1}{}_{010} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \tag{31}$$

$$R^{0}{}_{101} = -\Gamma^{0}_{01}\Gamma^{0}_{01} - \partial_1\Gamma^{0}_{01} = \frac{1}{4A^2}A_1^2 - \frac{1}{2A}A_{11} \tag{32}$$

$$R^{0}{}_{110} = -R^{0}{}_{101} = -\frac{1}{4A^2}A_1^2 + \frac{1}{2A}A_{11} \tag{33}$$

To get the Riemann tensor with all 4 indices lowered, we multiply by the metric:

$$R_{\epsilon\nu\lambda\sigma} = g_{\epsilon\mu}R^{\mu}{}_{\nu\lambda\sigma} \tag{34}$$

Here, the only two metric components we need are $g_{00} = -A$ and $g_{11} = 1$ so

$$R_{1010} = g_{11}R^{1}{}_{010} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \tag{35}$$

$$R_{1001} = g_{11}R^{1}{}_{001} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \tag{36}$$

$$R_{0101} = g_{00}R^{0}{}_{101} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \tag{37}$$

$$R_{0110} = g_{00}R^{0}{}_{110} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \tag{38}$$

Note that in this lowered form, the symmetries of the Riemann tensor are obeyed: $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$.

Finally, if we impose the condition 15 in the form $A_{11} = \frac{1}{2A}A_1^2$, we find that all four of these components are zero, thus making the entire Riemann tensor zero, indicating that spacetime is completely flat. [There are a lot of indices flying about here, so I'm hoping I got them all right...]