POLE IN A BARN PARADOX

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Along with the twin paradox, the pole in a barn paradox is probably the most famous of the so-called paradoxes arising from special relativity. There are various versions of the paradox, but the one we'll look at here is typical.

We are faced with a barn of length (in its rest frame $\mathcal{O}$, which we'll also take to be the rest frame of a farmer who is watching the runner) 15 m with a door at one end. A runner picks up a pole of length (in the rest frame $\mathcal{O}'$ of the runner) of 20 m. Somewhat unrealistically, we imagine that the runner runs towards the barn at a speed of $v = 0.8$. This gives a factor

$$\gamma = \frac{1}{\sqrt{1-0.8^2}} = \frac{5}{3} \tag{1}$$

At $t = t' = 0$, the front end of the pole is level with the door of the barn. We'll take $x = x' = 0$ at this event. Due to length contraction, the farmer measures the length of the pole to be

$$l_{F\text{ pole}} = \frac{20}{\gamma} = 12\text{ m} \tag{2}$$

Thus the farmer thinks the pole will fit inside the barn. However, to the runner, the barn has a contracted length of

$$l_{R\text{ barn}} = \frac{15}{\gamma} = 9\text{ m} \tag{3}$$

so in the runner's frame, the pole will *not* fit inside the barn.

The farmer, who believes the pole will fit, closes the door of the barn at the time when the front end of the pole collides with the back end of the barn. The pole and runner then come to rest, so its length is now measured to be 20 m by both the farmer and the runner. Thus we appear to have a 20 m pole inside a 15 m barn. Hence the paradox.

It's perhaps easiest to see what's going on by drawing a spacetime diagram, as in Fig. 1.

The diagram is drawn from the point of view of the farmer. In this frame the world lines of the ends of the barn are the two vertical lines given by

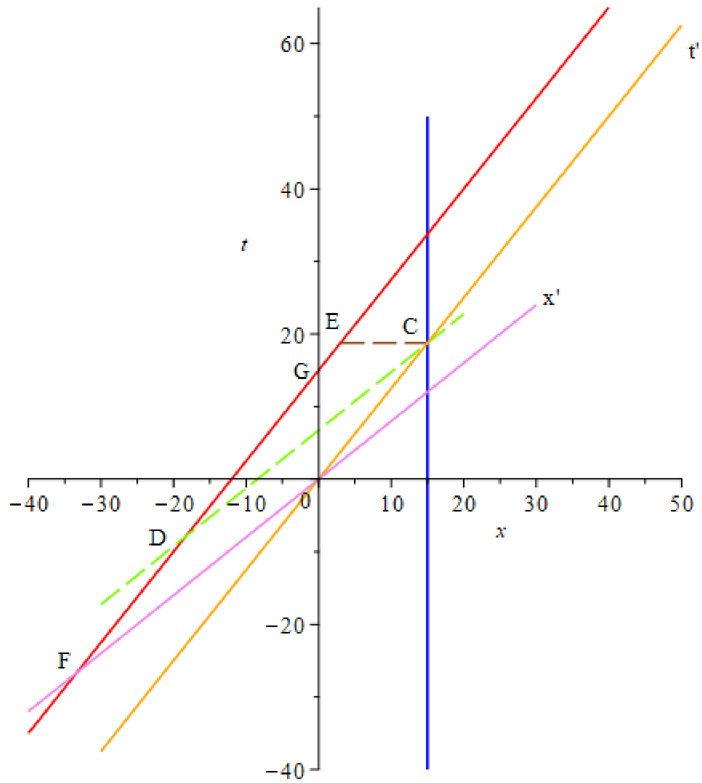

FIGURE 1. The pole in a barn paradox.

the t axis (for the end of the barn containing the door) and the blue line at $x = 15$. The runner's axes are the x' axis (with a slope of $v = 0.8$) shown in violet and the t' axis, with a slope of $\frac{1}{v} = \frac{5}{4}$, shown in orange.

The world lines of the ends of the pole are the t' axis for the front of the pole, and the line $t = \frac{x}{v} + 15$, shown in red, for the other end. The two ends of the pole as measured by the farmer are always at the same time t in the farmer's frame, and thus a horizontal line between the red and orange axes represents the pole as seen by the farmer. At $t = 0$, we see that the orange line is at the origin and the red line intersects the x axis at $x = -12$, thus giving the length of the pole as measured by the farmer as 12 m, as we saw above.

In the runner's frame, the two ends of the pole must be measured at the same time t', and thus must be connected by a line parallel to the x'axis. At $t = t' = 0$, the front of the pole is at $x = x' = 0$, so we can find the other end of the pole at $t' = 0$ by finding the intersection of the x' axis with the red line representing the back end of the pole. This intersection occurs at point F in the diagram. Thus the length of the pole in $\mathcal{O}'$ is the distance from F

to the origin. If this seems longer than 20 m remember that we must use the spacetime metric to measure distance, and not the Euclidean metric.

Now we consider the event when the front end of the pole collides with the back wall of the barn. This happens at point C in the diagram, which is the intersection of the world line of the front of the pole (orange line) with the world line of the back end of the barn (blue line). We can see that in frame $\mathcal{O}$ the back end of the pole is at point E which lies to the right of the world line of the front of the barn (which is the t axis), so according to the farmer, the entire pole *does* fit inside the barn at the event when the front of the pole collides with the back of the barn.

To the runner, however, we can determine the location of the back end of the pole by drawing a line from C parallel to the x' axis and finding where that line intersects the world line of the back end of the pole. This line is the dashed line in green, and we see that it intersects the world line of the back end of the pole (red line) at point D, which is *outside* the barn, as it's to the left of the t axis. Thus the runner says that when the pole collides with the back end of the barn, the pole does *not* fit inside the barn.

We can get some numerical values by using Lorentz transformations. The transformations from frame $\mathcal{O}'$ to $\mathcal{O}$ are

$$\begin{aligned} x &= \gamma\left(x'-vt'\right) \\ t &= \gamma\left(t'-vx'\right) \end{aligned} \tag{4}$$

where $v=-0.8$ since relative to the runner, the farmer is moving to the left at speed 0.8. The inverse transformations are found by replacing v by $-v$:

$$\begin{aligned} x' &= \gamma\left(x+vt\right) \\ t' &= \gamma\left(t+vx\right) \end{aligned} \tag{5}$$

The event where the front of the pole passes the door (point C in Fig. 1) of the barn is

$$(t,x)=\left(t',x'\right)=(0,0) \tag{6}$$

When the front of the pole reaches the back of the barn, $t=\frac{15}{v}=\frac{75}{4}$, and $x=15$. Thus the runner sees this event at

$$x'=\frac{5}{3}\left(15-\frac{4}{5}\times\frac{75}{4}\right)=0 \tag{7}$$

$$t'=\frac{5}{3}\left(\frac{75}{4}-\frac{4}{5}\times 15\right)=\frac{45}{4} \tag{8}$$

The event where the back end of the pole passes the open doorway and the farmer closes the door (as seen by $\mathcal{O}$; point G in Fig. 1) is $(t,x) = (15,0)$, since the time taken for the back end of the pole to reach the door, as seen by $\mathcal{O}$, is $t = \frac{12}{v} = 12\times\frac{5}{4} = 15$. Using 5, we find that in the $\mathcal{O}'$ system, the coordinates are $(t',x') = (25,-20)$.

The interval between these two events is

$$\begin{aligned}\Delta s^2 &= -\Delta t^2 + \Delta x^2 && (9)\\ &= -\left(\frac{75}{4} - 15\right)^2 + (15-0)^2 && (10)\\ &= \frac{3375}{16} && (11)\end{aligned}$$

We can verify this by calculating the interval in the $\mathcal{O}'$ system as

$$\begin{aligned}\Delta s'^2 &= -\Delta t'^2 + \Delta x'^2 && (12)\\ &= -\left(\frac{45}{4} - 25\right)^2 + (20)^2 && (13)\\ &= \frac{3375}{16} && (14)\end{aligned}$$

Since $\Delta s^2 > 0$, this is a spacelike interval, which means that there is always a non-zero distance between the two events, no matter what frame they are viewed from. However, it is possible to find a frame in which the two events are simultaneous.

As we can see from Fig. 1, there is never a time in $\mathcal{O}'$ when the entire pole is inside the barn. Imagine sliding the line segment CD parallel to the t' axis, and you can see that this segment will never fit between the t axis and the blue line. This makes sense from the runner's point of view, since he views the barn as shorter than the pole, so of course the pole will never fit.

From the point of view of the farmer, however, there is a time when the pole does fit inside the barn, but only while it is moving, so its length appears contracted. If we brought the pole to rest inside the barn after shutting the door, then both the runner and farmer would agree that its length is now 20 m and thus longer than the barn.

There has been a lot of discussion (see the Wikipedia article, for example) about how to resolve the paradox. The main conclusion is that, if we brought the pole to rest after locking it in the barn, the pole must undergo deceleration, so it's no longer in an inertial frame, and in fact is transitioning through a series of frames from travelling at $v = 0.8$ to rest. This is similar to the situation in the twin paradox, where there is a genuine difference between the ages of the twins due to one twin undergoing deceleration and re-acceleration when they turn around at their destination, while the other twin remains in an inertial frame throughout the process. Bringing the ladder to rest causes a real difference between the points of view of the farmer and the runner, so in that sense, the paradox is real. The farmer really does believe that the entire pole will fit into the barn at the same time, while the runner does not. The difference arises because the time at which the farmer measures the endpoints of the pole are not simultaneous to the runner, and vice versa.