PROJECTION OPERATOR IN SPACETIME

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Consider a timelike unit four-vector $\vec{U}$ and a tensor P with components

$$P_{\mu\nu}=\eta_{\mu\nu}+U_{\mu}U_{\nu} \tag{1}$$

P acts as a projection operator in the sense that applying it to an arbitrary vector $\vec{V}$ produces a vector $\vec{V}_{\perp}$ that is orthogonal to $\vec{U}$.

Since $\vec{U}$ is timelike and a unit vector, its square magnitude is -1:

$$\eta_{\mu\nu}U^{\mu}U^{\nu}=-1 \tag{2}$$

We therefore have

$$\begin{aligned} V_{\perp}^{\alpha} &\equiv P^{\alpha}{}_{\beta}V^{\beta} && (3)\\ &=\left(\eta^{\alpha}{}_{\beta}+U^{\alpha}U_{\beta}\right)V^{\beta} && (4)\\ &=V^{\alpha}+U^{\alpha}U_{\beta}V^{\beta} && (5)\end{aligned}$$

Taking the scalar product with $\vec{U}$ we have

$$\begin{aligned} \eta_{\alpha\gamma}V_{\perp}^{\alpha}U^{\gamma} &=\eta_{\alpha\gamma}V^{\alpha}U^{\gamma}+\eta_{\alpha\gamma}U^{\alpha}U_{\beta}V^{\beta}U^{\gamma} && (6)\\ &=V^{\alpha}U_{\alpha}+\left(\eta_{\alpha\gamma}U^{\alpha}U^{\gamma}\right)U_{\beta}V^{\beta} && (7)\\ &=V^{\alpha}U_{\alpha}-U_{\beta}V^{\beta} && (8)\\ &=0 && (9)\end{aligned}$$

where we used 2 to get the third line. Thus 5 is orthogonal to $\vec{U}$.

A projection operator should have the property that if it is applied twice, the second application has no further effect. We can verify this as follows.

$$\begin{aligned} P^{\alpha}{}_{\beta}V_{\perp}^{\beta} &=\left(\eta^{\alpha}{}_{\beta}+U^{\alpha}U_{\beta}\right)\left(V^{\beta}+U^{\beta}U_{\gamma}V^{\gamma}\right) && (10)\\ &=V^{\alpha}+U^{\alpha}U_{\gamma}V^{\gamma}+U^{\alpha}U_{\beta}V^{\beta}-U^{\alpha}U_{\gamma}V^{\gamma} && (11)\\ &=V^{\alpha}+U^{\alpha}U_{\gamma}V^{\gamma} && (12)\\ &=V_{\perp}^{\alpha} && (13)\end{aligned}$$

We can generalize this result from a timelike unit vector $\vec{U}$ to an arbitrary non-null vector $\vec{q}$. The projection operator is then

$$Q_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^\alpha q_\alpha} \tag{14}$$

If we apply this to some vector $\vec{V}$ we have

$$Q^\mu{}_\nu V^\nu = \left[\eta^\mu{}_\nu - \frac{q^\mu q_\nu}{q^\alpha q_\alpha}\right] V^\nu \tag{15}$$

$$= V^\mu - q^\mu \frac{q_\nu V^\nu}{q^\alpha q_\alpha} \tag{16}$$

Applying this back on $\vec{q}$ we have

$$Q^\mu{}_\nu V^\nu q_\mu = q_\mu V^\mu - q^\mu q_\mu \frac{q_\nu V^\nu}{q^\alpha q_\alpha} \tag{17}$$

$$= q_\mu V^\mu - q_\nu V^\nu \tag{18}$$

$$= 0 \tag{19}$$

This doesn't work for a null $\vec{q}$ since $q^\alpha q_\alpha = 0$ and we'd be dividing by 0 in 14.

Returning to $P_{\mu\nu}$ in 1, we can see that it acts like the metric tensor on any two vectors orthogonal to $\vec{U}$. We have

$$P\left(\vec{V}_\perp, \vec{W}_\perp\right) = \left(\eta_{\mu\nu} + U_\mu U_\nu\right) V_\perp^\mu W_\perp^\nu \tag{20}$$

$$= \eta_{\mu\nu} V_\perp^\mu W_\perp^\nu + U_\mu U_\nu V_\perp^\mu W_\perp^\nu \tag{21}$$

Since $\vec{V}_\perp$ and $\vec{W}_\perp$ are orthogonal to $\vec{U}$, the last term on the RHS is zero, and we have

$$P\left(\vec{V}_\perp, \vec{W}_\perp\right) = \eta_{\mu\nu} V_\perp^\mu W_\perp^\nu = \vec{V}_\perp \cdot \vec{W}_\perp \tag{22}$$