PROPER TIME TO FALL THROUGH THE EVENT HORIZON

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Continuing our exploration of the Schwarzschild metric and its behaviour at $r = 2GM$ (known as the event horizon), we'll look here at the proper time an object measures as it crosses the event horizon. The metric is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

Consider an object starting at rest at a radial coordinate $r = R$ and falling radially in towards the event horizon. To work out its proper time, we start with the equation of motion for r:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\ell^2}{r^2} - GM\left(\frac{1}{r} + \frac{\ell^2}{r^3}\right) = \frac{1}{2}\left(e^2 - 1\right) \tag{2}$$

Since the object is at rest initially, $dr/d\tau = 0$ and its angular momentum is $\ell = 0$ so we get for its energy

$$e = \sqrt{1 - \frac{2GM}{R}} \tag{3}$$

The energy e is a conserved quantity and the angular momentum remains at $\ell = 0$, so the equation of motion becomes

$$\frac{dr}{d\tau} = \sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R}} \tag{4}$$

We can integrate this to find the proper time that elapses as the object falls from $r = R$ to $r = 0$:

$$\int_R^0 \frac{-dr}{\sqrt{\frac{1}{r} - \frac{1}{R}}} = \sqrt{2GM}\Delta\tau \tag{5}$$

(the minus sign in the integrand accounts for the fact that r is decreasing).

Using Maple, the integral is

$$\int \frac{-dr}{\sqrt{\frac{1}{r}-\frac{1}{R}}} = \frac{\sqrt{R}}{2}\left[2\sqrt{r(R-r)}+R\arctan\left(\frac{R-2r}{2\sqrt{r(R-r)}}\right)\right] \tag{6}$$

Plugging in the limits, we get

$$\sqrt{2GM}\Delta\tau = \frac{\pi R^{3/2}}{2} \tag{7}$$

$$\Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}} \tag{8}$$

Thus the proper time for an object to fall to $R=0$, even if it crosses the event horizon $r=2GM$ on the way, is finite and well-defined.

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