RELATION BETWEEN ENERGY AND VELOCITY OF OBSERVER

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A useful relation between the energy of an object and the four-velocity of a moving observer can be obtained as follows. The four-momentum of an object is defined as

$$\boldsymbol{p}=(E,\vec{p})=(E,p_x,p_y,p_z) \tag{1}$$

where E is the object's energy and $\vec{p}$ is its 3-momentum.

If we consider objects that have a rest mass (so not photons), then in that object's rest frame we have

$$\boldsymbol{p}=(m,0,0,0) \tag{2}$$

If an observer moves with a four-velocity with respect this object, the four-velocity can be written as

$$\boldsymbol{u}=\left(\frac{dx^0}{d\tau},\frac{dx^1}{d\tau},\frac{dx^2}{d\tau},\frac{dx^3}{d\tau}\right) \tag{3}$$

where τ is the proper time in the frame in which the observer's four-velocity is measured to be $\boldsymbol{u}$.

In the observer's rest frame, $x^0=\tau$ and the spatial coordinates do not change, so

$$\boldsymbol{u}_{\text{rest}}=\left(\frac{d\tau}{d\tau},0,0,0\right)=(1,0,0,0) \tag{4}$$

The four-velocity thus obeys the rule

$$\boldsymbol{u}_{\text{rest}}\cdot\boldsymbol{u}_{\text{rest}}=\eta_{\mu\nu}u_{\text{rest}}^{\mu}u_{\text{rest}}^{\nu}=-1 \tag{5}$$

As this is a tensor equation, it is valid in all frames, so we have the general identity for four-velocity:

$$\boxed{\boldsymbol{u}\cdot\boldsymbol{u}=-1} \tag{6}$$

Example 1. We can verify this for the case of an object moving with speed v in the x^1 direction. In that case, its four-velocity is

$$\boldsymbol{u} = (\gamma, \gamma v, 0, 0) \tag{7}$$

so we have

$$\boldsymbol{u}\cdot\boldsymbol{u} = -\gamma^2 + \gamma^2 v^2 \tag{8}$$

$$= \frac{1}{1-v^2}\left(-1+v^2\right) \tag{9}$$

$$= -1 \tag{10}$$

Now suppose we have an observer in his rest frame observing an object with mass m moving with four-velocity $\boldsymbol{v}$. To the observer, the object's four-momentum is

$$\boldsymbol{p} = (\gamma m, \gamma m\vec{v}) \tag{11}$$

where $\vec{v}$ is the 3-velocity. In the observer's rest frame, the observed energy of the moving object is γm, which can also be written as

$$E = -\boldsymbol{u}_{\text{rest}}\cdot\boldsymbol{p} \tag{12}$$

$$= -(-1\times\gamma m) \tag{13}$$

$$= \gamma m \tag{14}$$

Again, this is a tensor equation so is valid in any frame. In particular, if the observer is moving with velocity $\boldsymbol{u}_{\text{obs}}$ and measures the four-momentum of an object as $\boldsymbol{p}$, then the energy as seen by the moving observer is

$$\boxed{E = -\boldsymbol{u}_{\text{obs}}\cdot\boldsymbol{p}} \tag{15}$$

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