RIEMANN NORMAL COORDINATES

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The local flatness theorem states that, in a given manifold, the metric $g_{\alpha\beta}$ at each point can be approximated by the flat metric $\eta_{\alpha\beta}$, and that the first derivatives of the components of $g_{\alpha\beta}$ are all zero. Another way of saying this is that, at a given point $\mathcal{P}$, an object moving along a geodesic moves in an approximately straight line. This is equivalent to the case of a continuous, differentiable mathematical function that is approximated by the first-order term in its Taylor expansion.

To formalize this, we construct an orthonormal basis $\{\boldsymbol{e}_\alpha\}$ at the point $\mathcal{P}$. At $\mathcal{P}$, there will in general be a number of geodesics that pass through $\mathcal{P}$ travelling in various directions. We can pick one of these directions and construct a unit vector $\boldsymbol{n}$ that points in that direction. For a small distance s, the coordinates of a point on the chosen geodesic can then be written as

$$x^\alpha = sn^\alpha \tag{1}$$

If $\boldsymbol{n}$ is a spacelike vector, then s is a spacelike distance and the fact that $\boldsymbol{n}$ is a unit vector is specified by the condition

$$\boldsymbol{n}\cdot\boldsymbol{n} = \eta_{\alpha\beta}n^\alpha n^\beta = +1 \tag{2}$$

If $\boldsymbol{n}$ is timelike, then we use the proper time τ so that

$$x^\alpha = \tau n^\alpha \tag{3}$$

and the unit vector condition is satisfied by

$$\boldsymbol{n}\cdot\boldsymbol{n} = \eta_{\alpha\beta}n^\alpha n^\beta = -1 \tag{4}$$

If $\boldsymbol{n}$ is lightlike, then we cannot make it into a unit vector, since all lightlike vectors are null, having zero length. However, we can still define $\boldsymbol{n}$ to be a null vector that points in the direction of a photon's geodesic. Null vectors can be thought of as vectors 'between' spacelike and timelike vectors. For example, a vector such as

$$\boldsymbol{x} = \left(1, 1, 10^{-4}, 0\right) \tag{5}$$

has a squared length of

$$\eta_{\alpha\beta}x^{\alpha}x^{\beta}=-1+1+10^{-8}=10^{-8}>0 \tag{6}$$

so it is spacelike, but only just. A vector like

$$\boldsymbol{y}=(1.0001,1,0,0) \tag{7}$$

has a squared length

$$\eta_{\alpha\beta}y^{\alpha}y^{\beta}=-1.00020001+1=-2.0001\times10^{-4}<0 \tag{8}$$

so it is timelike, again, only just. The vector

$$\boldsymbol{\ell}=(1,1,0,0) \tag{9}$$

has a length of zero and is lightlike. It can be seen to lie between $\boldsymbol{x}$ and $\boldsymbol{y}$ in the sense that it can be obtained by slightly varying the 0 and 2 components of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Example 1. If we consider flat space, then all geodesics are straight lines, so $\boldsymbol{n}$ could be any vector pointing in any direction. For the vector $\boldsymbol{x}$ in 5, we can define a unit vector as

$$\boldsymbol{n}_x=\left(10^4,10^4,1,0\right) \tag{10}$$

This has unit length since

$$\eta_{\alpha\beta}n_x^{\alpha}n_x^{\beta}=-10^4+10^4+1=1 \tag{11}$$

and is parallel to $\boldsymbol{x}$. For values of s, we therefore have

$$x^{\alpha}=sn^{\alpha} \tag{12}$$

The Riemann normal coordinates of a point at a distance s in the direction of $\boldsymbol{n}_x$ are therefore

$$sn^{\alpha}=\left(10^4s,10^4s,s,0\right) \tag{13}$$

For the vector $\boldsymbol{y}$ we can define $\boldsymbol{n}$ as

$$\boldsymbol{n}_y=a\boldsymbol{y} \tag{14}$$

such that $\boldsymbol{n}_y\cdot\boldsymbol{n}_y=-1$. That is

$$a^2\left(\boldsymbol{y}\cdot\boldsymbol{y}\right)=-2.0001\times10^{-4}a^2=-1 \tag{15}$$

which gives

$$a = \frac{1}{\sqrt{2.0001 \times 10^{-4}}} \tag{16}$$
$$\approx 70.7089 \tag{17}$$

Thus

$$\boldsymbol{n}_y = (70.7089\ldots \times 1.0001, 70.7089\ldots, 0, 0) \tag{18}$$
$$\approx (70.71598, 70.7089, 0, 0) \tag{19}$$

The Riemann normal coordinates of a point with proper time τ in the direction of $\boldsymbol{y}$ are then

$$\tau \boldsymbol{n}_y \approx (70.71598\tau, 70.7089\tau, 0, 0) \tag{20}$$

Along a null vector $\boldsymbol{n}$, the Riemann normal coordinates would be given by $s\boldsymbol{n}$.

Example 2. An example of Riemann normal coordinates in a curved space is provided by a sphere embedded in 3-d flat space. The metric in spherical coordinates is the usual

$$dS^2 = a^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{21}$$

At the north pole, all geodesics are lines of longitude, that is, lines with a constant ϕ. We can choose an orthonormal basis at the north pole by defining unit vectors in the directions of $\phi = 0$ and $\phi = \frac{\pi}{2}$. A unit vector in the direction specified by an arbitrary value of ϕ is then

$$\boldsymbol{n} = (\cos\phi, \sin\phi) \tag{22}$$

At the north pole, $\theta = 0$. The arc length along a line of latitude starting at the north pole is $a\theta$, so for small values of θ the displacement along a straight line path heading away from the north pole is

$$s \approx a\theta \tag{23}$$

The Riemann normal coordinates for the north pole are then

$$x^\alpha = sn^\alpha \tag{24}$$

or

$$\boldsymbol{x} = (a\theta\cos\phi, a\theta\sin\phi) \tag{25}$$

We can show that the metric at the north pole does actually become approximately the flat space metric by starting with 25, written as

$$(x,y) = (a\theta\cos\phi, a\theta\sin\phi) \tag{26}$$

Inverting the coordinates, we have

$$\theta = \frac{\sqrt{x^2+y^2}}{a} \tag{27}$$

$$\phi = \arctan\frac{y}{x} \tag{28}$$

The differentials that we need to insert into 21 are

$$d\theta = \frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}\,a} \tag{29}$$

$$d\phi = \frac{-y\ dx + x\ dy}{x^2+y^2} \tag{30}$$

Near the north pole, θ is small, so we can take $\sin\theta \approx \theta$ in 21. This gives

$$dS^2 = a^2\left(\frac{x\ dx + y\ dy}{\sqrt{x^2+y^2}\,a}\right)^2 + a^2\frac{\left(x^2+y^2\right)}{a^2}\left(\frac{-y\ dx + x\ dy}{x^2+y^2}\right)^2 \tag{31}$$

$$= dx^2 + dy^2 \tag{32}$$

where I used Maple to do the algebra to get the last line. Thus to leading order (using the approximation $\sin\theta \approx \theta$), we do indeed get the flat space metric.

A more accurate expression for the metric can be obtained by retaining the first two terms in the $\sin\theta$ expansion, that is, we use

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} \tag{33}$$

Using 27, this gives

$$\sin\theta \approx \frac{\sqrt{x^2+y^2}}{a} - \frac{1}{6}\left[\frac{\sqrt{x^2+y^2}}{a}\right]^3 \tag{34}$$

As you can see, this gets quite messy when inserted back into 21, but we can use Maple to do the algebra. Keeping terms up to second order in x and y, we have

$$dS^2 = \left(1 - \frac{y^2}{3a^2}\right)dx^2 + \frac{2}{3}\frac{xy}{a^2}dx\ dy + \left(1 - \frac{x^2}{3a^2}\right)dy^2 \tag{35}$$

The metric $g_{\alpha\beta}$ is therefore

$$\left[g_{\alpha\beta}\right] = \left[\begin{array}{cc} 1-\frac{y^2}{3a^2} & \frac{xy}{3a^2} \\ \frac{xy}{3a^2} & 1-\frac{x^2}{3a^2} \end{array}\right] \tag{36}$$

Note that all first derivatives of $g_{\alpha\beta}$ with respect to x and y are zero at $(x,y) = (0,0)$, and that at this point, the metric reverts to

$$\left[g_{\alpha\beta}(0,0)\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \tag{37}$$