

RIEMANN TENSOR FOR 3-D SPHERICAL COORDINATES

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We saw that the Riemann tensor for the surface of a sphere had a non-zero component, indicating that this is a curved space. If we use spherical coordinates in 3-d space, however, the Riemann tensor should be zero, since this is a flat space.

As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

$$\frac{d}{d\tau} (g_{aj} \dot{x}^j) - \frac{1}{2} \partial_a g_{ij} \dot{x}^i \dot{x}^j = 0 \quad (1)$$

$$\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0 \quad (2)$$

where as usual a dot denotes a derivative with respect to proper time τ .

For spherical coordinates, the interval is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (3)$$

Note that r is now a variable, rather than the constant radius of the sphere in the 2-d system.

From 1 we get, with $a = \theta$:

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \quad (4)$$

Dividing through by r^2 and comparing with 2 we get

$$\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta \quad (5)$$

$$\Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \frac{1}{r} \quad (6)$$

$$\Gamma_{\theta\phi}^{\theta} = \Gamma_{\phi\theta}^{\theta} = \Gamma_{\theta\theta}^{\theta} = 0 \quad (7)$$

With $a = \phi$ we have

$$2r \sin^2 \theta \dot{\phi} + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0 \quad (8)$$

$$\frac{2}{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \ddot{\phi} = 0 \quad (9)$$

$$\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta \quad (10)$$

$$\Gamma_{r\phi}^{\phi} = \Gamma_{\phi r}^{\phi} = \frac{1}{r} \quad (11)$$

$$\Gamma_{\theta\theta}^{\phi} = \Gamma_{\phi\phi}^{\phi} = 0 \quad (12)$$

For $a = r$, we have

$$\ddot{r} - \frac{1}{2} (2r\dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2) = 0 \quad (13)$$

$$\ddot{r} - r\dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 = 0 \quad (14)$$

$$\Gamma_{\theta\theta}^r = -r \quad (15)$$

$$\Gamma_{\phi\phi}^r = -r \sin^2 \theta \quad (16)$$

We can use these results to get the Riemann tensor.

$$R_{abcd} = g_{af} R_{bcd}^f \quad (17)$$

$$= g_{af} \left(\partial_c \Gamma_{db}^f - \partial_d \Gamma_{cb}^f + \Gamma_{db}^k \Gamma_{ck}^f - \Gamma_{cb}^k \Gamma_{kd}^f \right) \quad (18)$$

We can work out one of the components to see how the calculations go.

$$R_{\theta\phi\theta\phi} = g_{\theta f} R_{\phi\theta\phi}^f \quad (19)$$

$$= g_{\theta\theta} R_{\phi\theta\phi}^{\theta} \quad (20)$$

$$= r^2 \left(\partial_{\theta} \Gamma_{\phi\phi}^{\theta} - \partial_{\phi} \Gamma_{\theta\phi}^{\theta} + \Gamma_{\phi\phi}^k \Gamma_{\theta k}^{\theta} - \Gamma_{\theta\phi}^k \Gamma_{k\phi}^{\theta} \right) \quad (21)$$

$$= r^2 \left(\sin^2 \theta - \cos^2 \theta - 0 - \frac{r \sin^2 \theta}{r} + \cos^2 \theta \right) \quad (22)$$

$$= 0 \quad (23)$$

The other components of the Riemann tensor can be evaluated in the same way, and they all come out to zero, so spherical coordinates represent flat 3-d space.