RIEMANN TENSOR FROM PARALLEL TRANSPORT

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We've seen how the Riemann curvature tensor arises from a consideration of geodesic deviation, which is the treatment given in Moore's book. Schutz gives a derivation based on the parallel transport of a vector around a closed loop in a manifold. This derivation is a bit easier to understand (at least for me), so it's worth going through it.

We consider a closed loop with four sides. Two of the sides are defined by lines of constant x^1 and the other two by lines of constant x^2, where the superscript denotes which coordinate we're considering (they aren't exponents, in other words).

We label the corners of the loop by A,B,C,D. The point A is where the lines $x^1=a$ and $x^2=b$ (a,b are constants) intersect. We then travel along the line AB, which is a line of constant $x^2=b$. Point B is the intersection of $x^1=a+\delta a$ and $x^2=b$. Then we travel along BC which is a line of constant $x^1=a+\delta a$, arriving at point C, which is the intersection of $x^1=a+\delta a$ and $x^2=b+\delta b$. Then we travel along CD, arriving at D which is where $x^1=a$ intersects $x^2=b+\delta b$. Finally, we travel along DA, arriving back at our starting point.

To help visualize this, recall the example of the parallel transport of a vector around a spherical triangle. Because the surface of a sphere is a curved space, parallel transport of a vector around a spherical triangle on the surface of the sphere results in the final orientation of the vector being different from its starting orientation. The idea here is to generalize this notion and *define* the curvature of a manifold by measuring how much the starting and final orientations of a vector differ after parallel transport around a loop.

The condition for parallel transport of a vector $\vec{V}$ is that its covariant derivative is zero along the chosen path. Written using Christoffel symbols, the condition is

$$\nabla_{\mu}V^{\nu}\equiv V^{\nu}{}_{;\mu}=V^{\nu}{}_{,\mu}+V^{\sigma}\Gamma^{\nu}{}_{\sigma\mu}=0 \tag{1}$$

Remember that the semicolon in the subscript $V^{\nu}{}_{;\mu}$ denotes the covariant derivative, while the comma in $V^{\nu}{}_{,\mu}$ denotes the partial derivative with respect to a given coordinate.

Now consider the first leg of our journey, from A to B. Along this edge, $x^2 = b =$ constant, so we can write

$$V^{\alpha}{}_{,1} = \frac{\partial V^{\alpha}}{\partial x^1} = -\Gamma^{\alpha}{}_{\mu 1} V^{\mu} \tag{2}$$

We can find $V^{\alpha}(B)$ by integrating this expression between points A and B. We have

$$V^{\alpha}(B) = V^{\alpha}(A_0) + \int_A^B \frac{\partial V^{\alpha}}{\partial x^1} dx^1 \tag{3}$$

$$= V^{\alpha}(A_0) - \int_A^B \Gamma^{\alpha}{}_{\mu 1} V^{\mu} dx^1 \tag{4}$$

where $V^{\alpha}(A_0)$ denotes the vector component V^{α} in its initial orientation. Due to the curvature of the space, we expect V^{α} to be different when we arrive back at point A after traversing the loop.

We can apply the same reasoning to move from B to C, along which $x^1 = a + \delta a =$ constant and x^2 runs between b and $b + \delta b$. Along this path we have

$$V^{\alpha}{}_{,2} = \frac{\partial V^{\alpha}}{\partial x^2} = -\Gamma^{\alpha}{}_{\mu 2} V^{\mu} \tag{5}$$

so we have

$$V^{\alpha}(C) = V^{\alpha}(B) - \int_B^C \Gamma^{\alpha}{}_{\mu 2} V^{\mu} dx^2 \tag{6}$$

Moving along C to D, however, *reduces* x^1 from $a + \delta a$ back down to a, so dx^1 must be replaced by $-dx^1$ in the integral. As a result, we have

$$V^{\alpha}(D) = V^{\alpha}(C) + \int_C^D \Gamma^{\alpha}{}_{\mu 1} V^{\mu} dx^1 \tag{7}$$

Note that the limits on the integral are from C to D, but we've replaced dx^1 by $-dx^1$, which is the reason for the + sign in front of the integral, as opposed to the minus in 4 and 6.

Finally, we travel from D to A, along which x^2 decreases from $b + \delta b$ back down to b, so again we have a negative differential. We end up with

$$V^{\alpha}(A_1) = V^{\alpha}(D) + \int_A^D \Gamma^{\alpha}{}_{\mu 2} V^{\mu} dx^2 \tag{8}$$

where $V^{\alpha}(A_1)$ is the vector component V^{α} *after* traversing the loop. As noted above, in general $V^{\alpha}(A_1) \neq V^{\alpha}(A_0)$.

We're interested in how much the vector changes after traversing the loop, so we need to calculate $\delta V^{\alpha}=V^{\alpha}(A_1)-V^{\alpha}(A_0)$. We can find this by substituting 4 into 6, then 6 into 7 and finally 7 into 8. That is, we have

$$V^{\alpha}(A_1)=V^{\alpha}(A_0)-\int_A^B \Gamma^{\alpha}{}_{\mu 1}V^{\mu}dx^1-\int_B^C \Gamma^{\alpha}{}_{\mu 2}V^{\mu}dx^2+\int_C^D \Gamma^{\alpha}{}_{\mu 1}V^{\mu}dx^1+\int_A^D \Gamma^{\alpha}{}_{\mu 2}V^{\mu}dx^2 \tag{9}$$

We note now that the limits on the two integrals over x^1 are actually the same (from $x^1=a$ to $x^1=a+\delta a$), but the value of x^2 is different in the two integrals. That is, we have

$$\int_C^D \Gamma^{\alpha}{}_{\mu 1}V^{\mu}dx^1-\int_A^B \Gamma^{\alpha}{}_{\mu 1}V^{\mu}dx^1=\int_a^{a+\delta a}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\big|_{x^2=b+\delta b}-\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\big|_{x^2=b}\right)dx^1 \tag{10}$$

Similarly for the other two integrals (over x^2), we have

$$\int_A^D \Gamma^{\alpha}{}_{\mu 2}V^{\mu}dx^2-\int_B^C \Gamma^{\alpha}{}_{\mu 2}V^{\mu}dx^2=\int_b^{b+\delta b}\left(-\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\big|_{x^1=a+\delta a}+\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\big|_{x^1=a}\right)dx^2 \tag{11}$$

So far, everything we've done has been exact, without assuming that the loop is infinitesimal. What we're really interested in, though, is the definition of the curvature of a manifold at a specific point, so we now go over to the case where the loop shrinks down to become infinitesimal. In that case, the integrands in each of these two integrals can be approximated by the first order term in a Taylor expansion. We have

$$\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\big|_{x^1=a+\delta a}=\left[\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right]_{x^1=a}+\left[\frac{\partial}{\partial x^1}\left(\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right)\right]_{x^1=a}\delta a \tag{12}$$

$$\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\big|_{x^2=b+\delta b}=\left[\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right]_{x^2=b}+\left[\frac{\partial}{\partial x^2}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right)\right]_{x^2=b}\delta b \tag{13}$$

The integrals now become

$$\int_a^{a+\delta a}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\big|_{x^2=b+\delta b}-\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\big|_{x^2=b}\right)dx^1=\delta b\int_a^{a+\delta a}\left[\frac{\partial}{\partial x^2}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right)\right]_{x^2=b}dx^1 \tag{14}$$

$$\int_b^{b+\delta b}\left(-\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\big|_{x^1=a+\delta a}+\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\big|_{x^1=a}\right)dx^2=-\delta a\int_b^{b+\delta b}\left[\frac{\partial}{\partial x^1}\left(\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right)\right]_{x^1=a}dx^2 \tag{15}$$

Each of these integrals on the RHS is over an infinitesimal interval, so we can, to first order, approximate them by multiplying the integrand by the interval. That is, we have

$$\delta b\int_{a}^{a+\delta a}\left[\frac{\partial}{\partial x^{2}}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right)\right]_{x^{2}=b}dx^{1}\approx\delta b\delta a\left[\frac{\partial}{\partial x^{2}}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right)\right]_{x^{2}=b} \tag{16}$$

$$\delta a\int_{b}^{b+\delta b}\left[\frac{\partial}{\partial x^{2}}\left(\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right)\right]_{x^{1}=a}dx^{2}\approx\delta b\delta a\left[\frac{\partial}{\partial x^{1}}\left(\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right)\right]_{x^{1}=a} \tag{17}$$

The derivatives on the RHS involve derivatives of the Christoffel symbols, and of the vector components. The latter can be replaced by Christoffel symbols by using 2. We have

$$\frac{\partial}{\partial x^{2}}\left(\Gamma^{\alpha}{}_{\mu 1}V^{\mu}\right)=\Gamma^{\alpha}{}_{\mu 1,2}V^{\mu}+\Gamma^{\alpha}{}_{\mu 1}V^{\mu}{}_{,2} \tag{18}$$

$$=\Gamma^{\alpha}{}_{\mu 1,2}V^{\mu}+\Gamma^{\alpha}{}_{\mu 1}\left(-\Gamma^{\mu}{}_{\nu 2}V^{\nu}\right) \tag{19}$$

$$=\left(\Gamma^{\alpha}{}_{\mu 1,2}-\Gamma^{\alpha}{}_{\nu 1}\Gamma^{\nu}{}_{\mu 2}\right)V^{\mu} \tag{20}$$

where we swapped indices $\mu\leftrightarrow\nu$ in the last term to get the third line from the second. Similarly, we have

$$-\frac{\partial}{\partial x^{1}}\left(\Gamma^{\alpha}{}_{\mu 2}V^{\mu}\right)=\left(-\Gamma^{\alpha}{}_{\mu 2,1}+\Gamma^{\alpha}{}_{\nu 2}\Gamma^{\nu}{}_{\mu 1}\right)V^{\mu} \tag{21}$$

Putting it all together, we have

$$\delta V^{\alpha}=\delta a\delta b\left[\Gamma^{\alpha}{}_{\mu 1,2}-\Gamma^{\alpha}{}_{\mu 2,1}+\Gamma^{\alpha}{}_{\nu 2}\Gamma^{\nu}{}_{\mu 1}-\Gamma^{\alpha}{}_{\nu 1}\Gamma^{\nu}{}_{\mu 2}\right]V^{\mu} \tag{22}$$

There was nothing special about the coordinates x^1 and x^2, so we can replace the indexes 1 and 2 by general indices β and σ to get

$$\delta V^{\alpha}=\delta a\delta b\left[\Gamma^{\alpha}{}_{\mu\beta,\sigma}-\Gamma^{\alpha}{}_{\mu\sigma,\beta}+\Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\nu}{}_{\mu\beta}-\Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\nu}{}_{\mu\sigma}\right]V^{\mu} \tag{23}$$

Since δV^{α} and V^{μ} are tensors (vectors) and δa and δb are just numbers (scalars), the object in brackets must be a tensor of rank $\binom{1}{3}$. This is the Riemann curvature tensor:

$$\boxed{R^{\alpha}{}_{\mu\sigma\beta}=\Gamma^{\alpha}{}_{\mu\beta,\sigma}-\Gamma^{\alpha}{}_{\mu\sigma,\beta}+\Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\nu}{}_{\mu\beta}-\Gamma^{\alpha}{}_{\nu\beta}\Gamma^{\nu}{}_{\mu\sigma}} \tag{24}$$

As noted in the earlier derivation, some books use the negative of this expression, but fortunately both Moore and Schutz use the same sign convention.