RIEMANN TENSOR IN 2-D FLAT SPACE

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Post date: 23 Jan 2023.

Another example of the Riemann tensor in a 2-d space. The metric is given as

$$ds^2 = dp^2 + \frac{dq^2}{b^2q^2} \tag{1}$$

where b is a constant. The metric tensor is therefore $g_{pp} = 1$, $g_{qq} = 1/b^2q^2$. By comparing the two forms of the geodesic equation, we can calculate the Christoffel symbols.

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \tag{2}$$

$$\ddot{x}^a + \Gamma^a_{ij}\dot{x}^j\dot{x}^i = 0 \tag{3}$$

With $a = p$, we get from 2

$$\ddot{p} = 0 \tag{4}$$

From 3, we see that $\Gamma^p_{ij} = 0$ for all i and j.

With $a = q$, we have

$$\frac{1}{b^2q^2}\ddot{q} - \frac{2}{b^2q^3}\dot{q}^2 + \frac{1}{b^2q^3}\dot{q}^2 = 0 \tag{5}$$

$$\ddot{q} - \frac{1}{q}\dot{q}^2 = 0 \tag{6}$$

Comparing with 3 we find

$$\Gamma^q_{qq} = -\frac{1}{q} \tag{7}$$

$$\Gamma^q_{pq} = \Gamma^q_{qp} = \Gamma^q_{pp} = 0 \tag{8}$$

The only independent component of the Riemann tensor in 2-d is R^p_{qpq} :

$$R^p_{\ qpq} = \partial_p \Gamma^p_{\ qq} - \partial_q \Gamma^p_{\ pq} + \Gamma^p_{\ kp}\Gamma^k_{\ qq} - \Gamma^p_{\ qk}\Gamma^k_{\ pq} \tag{9}$$

$$= 0 \tag{10}$$

since all terms involve components of the form Γ^p_{ij}. Therefore, the Ricci tensor is also zero: $R_{ij} = 0$ as is the curvature scalar $R = 0$. The space is flat.