

RIEMANN TENSOR IN 2-D POLAR COORDINATES

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As a simple example of the Riemann tensor we can work out one of its components in 2-d polar coordinates. The tensor is

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km} \quad (1)$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj} \ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2} \partial_a g_{ij} \right) \dot{x}^j \dot{x}^i = 0 \quad (2)$$

$$\ddot{x}^m + \Gamma^m{}_{ij} \dot{x}^j \dot{x}^i = 0 \quad (3)$$

The metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (4)$$

so $g_{rr} = 1$ and $g_{\theta\theta} = r^2$. For the two coordinates, 2 gives us

$$\ddot{r} - r\dot{\theta}^2 = 0 \quad (5)$$

$$r^2 \ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \quad (6)$$

Comparing with 3 we get

$$\Gamma^r{}_{\theta\theta} = -r \quad (7)$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} \quad (8)$$

One component of the Riemann tensor is :

$$R^r{}_{\theta r \theta} = \partial_r \Gamma^r{}_{\theta\theta} - \partial_\theta \Gamma^r{}_{r\theta} + \Gamma^r{}_{kr} \Gamma^k{}_{\theta\theta} - \Gamma^r{}_{\theta k} \Gamma^k{}_{r\theta} \quad (9)$$

$$= -1 - 0 + 0 + 1 \quad (10)$$

$$= 0 \quad (11)$$

In fact, from the symmetries of R , there is only one independent component of R in a 2-d metric, so all other components can be written as multiples of $R^r_{\theta r \theta}$. Thus for polar coordinates $R^r_{\theta r \theta} = 0$ implies all components of R are zero, which is what we'd expect since polar coordinates describe flat space.