RIEMANN TENSOR IN AN EXPONENTIAL 2-D CURVED SPACE

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Post date: 23 Jan 2023.

Here's another example of the Riemann tensor in a 2-d coordinate system. The tensor is

$$R^i_{\ j\ell m} \equiv \partial_\ell \Gamma^i_{\ mj} - \partial_m \Gamma^i_{\ \ell j} + \Gamma^k_{\ mj}\Gamma^i_{\ \ell k} - \Gamma^k_{\ \ell j}\Gamma^i_{\ km} \tag{1}$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \tag{2}$$

$$\ddot{x}^a + \Gamma^a_{\ ij}\dot{x}^j\dot{x}^i = 0 \tag{3}$$

The metric is

$$ds^2 = dp^2 + e^{2p/p_0}dq^2 \tag{4}$$

so $g_{pp} = 1$ and $g_{qq} = e^{2p/p_0}$. For the two coordinates, 2 gives us

$$\ddot{p} - \frac{1}{p_0}e^{2p/p_0}\dot{q}^2 = 0 \tag{5}$$

$$e^{2p/p_0}\ddot{q} + \frac{2}{p_0}e^{2p/p_0}\dot{p}\dot{q} = 0 \tag{6}$$

Dividing through by the coefficient of the second derivative in the second equation case gives:

$$\ddot{p} - \frac{1}{p_0}e^{2p/p_0}\dot{q}^2 = 0 \tag{7}$$

$$\ddot{q} + \frac{2}{p_0}\dot{p}\dot{q} = 0 \tag{8}$$

Comparing with 3 we get

$$\Gamma^{p}_{\ qq} = -\frac{1}{p_0} e^{2p/p_0} \tag{9}$$

$$\Gamma^{q}_{\ pq} = \Gamma^{q}_{\ qp} = \frac{1}{p_0} \tag{10}$$

with all other Christoffel symbols equal to zero.

The only independent Riemann tensor component in 2-d is $R^{p}_{\ qpq}$:

$$\begin{aligned} R^{p}_{\ qpq} &= \partial_p \Gamma^{p}_{\ qq} - \partial_q \Gamma^{p}_{\ pq} + \Gamma^{p}_{\ kp} \Gamma^{k}_{\ qq} - \Gamma^{p}_{\ qk} \Gamma^{k}_{\ pq} && (11) \\ &= \partial_p \Gamma^{p}_{\ qq} - 0 + 0 - \Gamma^{p}_{\ qq} \Gamma^{q}_{\ pq} && (12) \\ &= -\frac{2}{p_0^2} e^{2p/p_0} + \frac{1}{p_0^2} e^{2p/p_0} && (13) \\ &= -\frac{1}{p_0^2} e^{2p/p_0} && (14) \end{aligned}$$

Any non-zero component indicates that the space is curved, so this metric represents a curved space.