

RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

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We'll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km} \quad (1)$$

As usual, we need the Christoffel symbols, but we've already worked these out.

$$\Gamma^t{}_{ij} = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

$$\Gamma^r{}_{ij} = \begin{bmatrix} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r \left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \left(1 - \frac{2GM}{r}\right) \end{bmatrix} \quad (3)$$

$$\Gamma^\theta{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{bmatrix} \quad (4)$$

$$\Gamma^\phi{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{bmatrix} \quad (5)$$

We can plug these into the formula above to get $R^t{}_{rtr}$. We have

$$R^t{}_{rtr} = \partial_t \Gamma^t{}_{rr} - \partial_r \Gamma^t{}_{tr} + \Gamma^k{}_{rr} \Gamma^t{}_{kt} - \Gamma^k{}_{tr} \Gamma^t{}_{rk} \quad (6)$$

We can work out these terms one at a time (only the index k is summed):

$$\partial_t \Gamma_{rr}^t = 0 \quad (7)$$

$$-\partial_r \Gamma_{tr}^t = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-2} \left(\frac{2GM}{r^2}\right) \quad (8)$$

$$= \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} + \frac{2G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2} \quad (9)$$

$$\Gamma_{rr}^k \Gamma_{kt}^t = \Gamma_{rr}^r \Gamma_{rt}^t \quad (10)$$

$$= -\frac{G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2} \quad (11)$$

$$-\Gamma_{tr}^k \Gamma_{rk}^t = -(\Gamma_{rt}^t)^2 \quad (12)$$

$$= -\frac{G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2} \quad (13)$$

Adding these up we get

$$R_{rtr}^t = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} \quad (14)$$

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \rightarrow \infty$, $R_{rtr}^t \rightarrow 0$ so the further we get from the mass, the less curved the spacetime becomes.

PINGBACKS

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