

## SCHWARZSCHILD METRIC - FOUR-MOMENTUM OF A PHOTON

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This is a first example of the use of the time component of the Schwarzschild metric. This metric is, for a spherical mass  $M$ :

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

Suppose we have an observer at a Schwarzschild radius  $R$  from the centre of a star of mass  $M$ , and this observer watches a photon move radially outward. The observer measures the energy of the photon to be  $E$ . We can use this to calculate the four-momentum of the photon.

In special relativity, for an observer at rest the observer's four-velocity is  $u^i = [1, 0, 0, 0]$  so the scalar product of the observer's four-velocity with another object's momentum (as measured by the observer) is

$$\mathbf{p} \cdot \mathbf{u}_{obs} = g_{ij} p^i u^j \quad (2)$$

$$= -p^t u^t \quad (3)$$

$$= -p^t \quad (4)$$

$$= -E \quad (5)$$

since the time component of an object's four-momentum is its energy. Since this is a tensor equation, it should be true in curved space-time as well. In the Schwarzschild metric, an observer at rest has

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 \quad (6)$$

The relation between  $ds$  and the proper time interval  $d\tau$  is, assuming that it's the same as in special relativity

$$ds^2 = -d\tau^2 \quad (7)$$

and the time component of the four-velocity is

$$u^t = \frac{dt}{d\tau} \quad (8)$$

so we have

$$u^t = \left[ \left(1 - \frac{2GM}{R}\right)^{-1/2}, 0, 0, 0 \right] \quad (9)$$

Therefore, we get

$$\mathbf{p} \cdot \mathbf{u}_{obs} = g_{ij} p^i u^j \quad (10)$$

$$= - \left(1 - \frac{2GM}{R}\right) p^t \left(1 - \frac{2GM}{R}\right)^{-1/2} \quad (11)$$

$$= -p^t \left(1 - \frac{2GM}{R}\right)^{1/2} \quad (12)$$

$$= -E \quad (13)$$

$$p^t = E \left(1 - \frac{2GM}{R}\right)^{-1/2} \quad (14)$$

For a photon,  $\mathbf{p} \cdot \mathbf{p} = 0$ , and for a photon moving in the radial direction  $p^\theta = p^\phi = 0$  so

$$\mathbf{p} \cdot \mathbf{p} = g_{ij} p^i p^j \quad (15)$$

$$0 = - \left(1 - \frac{2GM}{R}\right) E^2 \left(1 - \frac{2GM}{R}\right)^{-1} + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 \quad (16)$$

$$0 = -E^2 + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 \quad (17)$$

$$p^r = E \sqrt{1 - \frac{2GM}{R}} \quad (18)$$

Thus the photon's four-momentum in the Schwarzschild basis is

$$\mathbf{p} = \left[ E \left(1 - \frac{2GM}{R}\right)^{-1/2}, E \sqrt{1 - \frac{2GM}{R}}, 0, 0 \right] \quad (19)$$

This formula is valid only if  $R > 2GM$  (the Schwarzschild radius), since if  $R < 2GM$ , the square roots become imaginary. As  $R \rightarrow \infty$ , the formula reverts to the relation for flat space  $\mathbf{p} \rightarrow [E, E, 0, 0]$ . As we get very far from the mass, its effect becomes insignificant.

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