SCHWARZSCHILD METRIC - FOUR-MOMENTUM OF A PHOTON

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This is a first example of the use of the time component of the Schwarzschild metric. This metric is, for a spherical mass M:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

Suppose we have an observer at a Schwarzschild radius R from the centre of a star of mass M, and this observer watches a photon move radially outward. The observer measures the energy of the photon to be E. We can use this to calculate the four-momentum of the photon.

In special relativity, for an observer at rest the observer's four-velocity is $u^i = [1,0,0,0]$ so the scalar product of the observer's four-velocity with another object's momentum (as measured by the observer) is

$$\mathbf{p}\cdot\mathbf{u}_{obs} = g_{ij}p^i u^j \tag{2}$$
$$= -p^t u^t \tag{3}$$
$$= -p^t \tag{4}$$
$$= -E \tag{5}$$

since the time component of an object's four-momentum is its energy. Since this is a tensor equation, it should be true in curved space-time as well. In the Schwarzschild metric, an observer at rest has

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 \tag{6}$$

The relation between ds and the proper time interval $d\tau$ is, assuming that it's the same as in special relativity

$$ds^2 = -d\tau^2 \tag{7}$$

and the time component of the four-velocity is

$$u^t = \frac{dt}{d\tau} \tag{8}$$

so we have

$$u^t = \left[\left(1 - \frac{2GM}{R}\right)^{-1/2}, 0, 0, 0\right] \tag{9}$$

Therefore, we get

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{u}_{obs} &= g_{ij} p^i u^j &(10)\\
&= -\left(1 - \frac{2GM}{R}\right) p^t \left(1 - \frac{2GM}{R}\right)^{-1/2} &(11)\\
&= -p^t \left(1 - \frac{2GM}{R}\right)^{1/2} &(12)\\
&= -E &(13)\\
p^t &= E\left(1 - \frac{2GM}{R}\right)^{-1/2} &(14)
\end{aligned}$$

For a photon, $\mathbf{p} \cdot \mathbf{p} = 0$, and for a photon moving in the radial direction $p^\theta = p^\phi = 0$ so

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{p} &= g_{ij} p^i p^j &(15)\\
0 &= -\left(1 - \frac{2GM}{R}\right) E^2 \left(1 - \frac{2GM}{R}\right)^{-1} + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 &(16)\\
0 &= -E^2 + \left(1 - \frac{2GM}{R}\right)^{-1} (p^r)^2 &(17)\\
p^r &= E\sqrt{1 - \frac{2GM}{R}} &(18)
\end{aligned}$$

Thus the photon's four-momentum in the Schwarzschild basis is

$$\mathbf{p} = \left[E\left(1 - \frac{2GM}{R}\right)^{-1/2}, E\sqrt{1 - \frac{2GM}{R}}, 0, 0\right] \tag{19}$$

This formula is valid only if $R > 2GM$ (the Schwarzschild radius), since if $R < 2GM$, the square roots become imaginary. As $R \to \infty$, the formula reverts to the relation for flat space $\boldsymbol{p} \to [E, E, 0, 0]$. As we get very far from the mass, its effect becomes insignificant.

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